

10 Interaction Between Electrons

This chapter deals with exploring the generic interaction types, which occur between the valence electrons of a metal.

10.1 Overview:

The matrix element of the potential energy between two electrons is defined by the Fourier transformed

$$V(\vec{k}, \vec{k}') = \frac{1}{V_2} \int d^3r_1 \int d^3r_2 V(\vec{r}_1, \vec{r}_2) e^{-i(\vec{k} - \vec{k}')(\vec{r}_1 - \vec{r}_2)} \quad (10.1)$$

As the potential energy $V(\vec{r}_1, \vec{r}_2)$ depends according to the Newton "action = - reaction" axiom (9.28) on the coordinate difference $\vec{r}_1 - \vec{r}_2$, the matrix element (10.1) turns out to depend only on the difference of the involved wave vectors:

$$V(\vec{k}, \vec{k}') = V(\vec{k} - \vec{k}') \quad (10.2)$$

Such a matrix element describes the scattering of one Cooper pair $(\vec{k}', -\vec{k}')$ to another Cooper pair $(\vec{k}, -\vec{k})$. The larger $V(\vec{k}, \vec{k}')$ is, the larger this scattering process is

$$V(\vec{k}, \vec{k}') : (\vec{k}', -\vec{k}') \rightarrow (\vec{k}, -\vec{k}) \quad (10.3)$$

In general one can distinguish two types of interaction. They are briefly introduced here and will be described in more detail in the later sections:

- 1) The Coulomb interaction acts directly between two electrons, which is repulsive. But, due to the presence of the atomic ion cores of the metal, the Coulomb interaction is partially screened.
- 2) Additionally, also an indirect interaction between two electrons is generated due to the virtual exchange of Landau quasi-particles, which can be attractive. Typical examples for such quasi-particles are provided by phonons or excitons.

The total interaction between two electrons is then additive in those two different contributions:

$$V = V_C + V_{e-QP} \quad (10.4)$$

Depending on which of both contributions is larger, a metal is either a superconductor ($V_C < V_{e-QP}$) or a normal conductor ($V_C > V_{e-QP}$).

The precious metals and the alkali metals are characterized by a large electric conductivity, i.e. a low electric resistance. Thus, the scattering of electrons with phonons is small. Therefore, the resulting total interaction (10.4) is positive and these metals are bad superconductors.

10.2 Screening of Coulomb Interaction:

At first we work out how the Coulomb interaction between two valence electrons is screened by the presence of inner electrons.

10.2.1 Jellium Model:

Within the jellium model of a metal it is assumed that the charge density of the conic atomic cores is uniformly smeared. Thus the valence electrons move practically freely through the metal. Denoting the constant, positive background charge density by ρ_I and representing the constant number density of valence electrons by n_0 , the charge neutrality of a metal implies in the jellium model

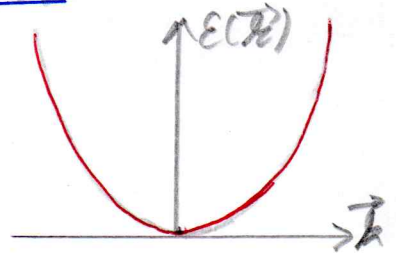
$$\rho_I = e n_0 \quad (10.5)$$

These assumptions of the jellium model are, in particular, justified for the alkali metals as there the valence electrons are only weakly bound due to the inner closed noble gas shell. Thus, the valence electrons are only weakly localised in the lattice of the solid.

10.2.2 Thomas-Fermi Approximation:

A free electron with a vanishing potential has according to the free electron gas model a parabolic dispersion:

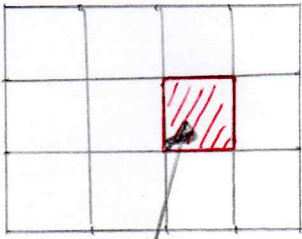
$$E(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2m} \quad (10.6)$$



Within a real lattice, however, the potential energy of an electron no longer vanishes. Provided that the potential energy varies spatially only finitely, it can be assumed to be approximately constant within a larger spatial region. Then there the Thomas-Fermi approximation is applicable, so at each space point locally the free

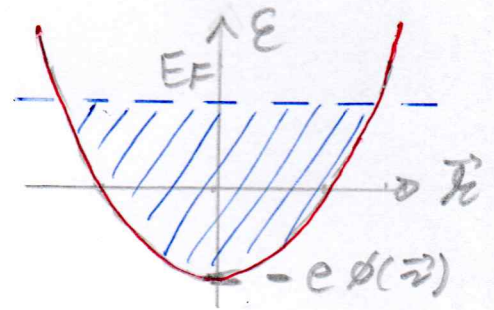
electron gas model is valid. Denoting the local electrostatic potential for an electron with $\phi(\vec{r})$, yields then the classical Hamiltonian

$$E(\vec{k}, \vec{r}) = \frac{\hbar^2 \vec{k}^2}{2m} - e\phi(\vec{r}) \quad (10.7)$$



locally constant potential at \vec{r}

Thomas-Fermi approximation \rightarrow



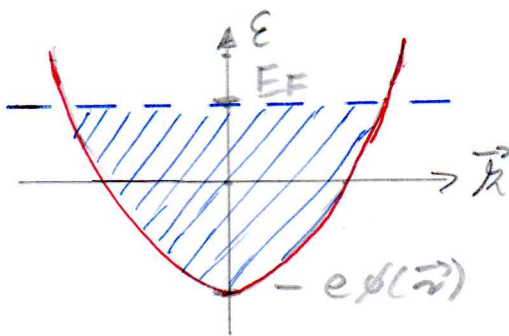
locally shifted dispersion relation

10.2.3 Number of Electrons in Fermi Sea:

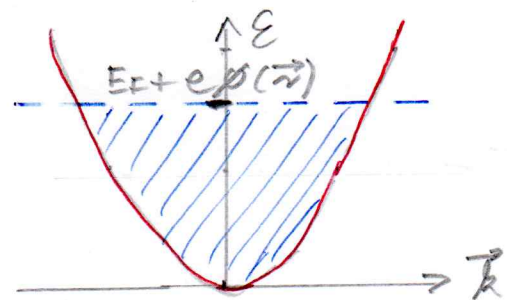
All these locally shifted dispersion relations (10.7) are filled in thermodynamic equilibrium up to the Fermi energy E_F . Thus, the density of electrons in the Fermi sea is given by

$$n_F(\vec{r}) = \int_{-e\phi(\vec{r})}^{E_F} dE N(E, \vec{r}) \quad (10.8)$$

Here $N(E, \vec{r})$ stands for the density of states of electrons at point \vec{r} and at energy E . But now the number of electrons in the interval $[-e\phi(\vec{r}), E_F]$ within the shifted dispersion relation (10.7) equals to the number of electrons in the interval $[0, E_F + e\phi(\vec{r})]$ within the unshifted dispersion relation (10.6):



density of states $N(E, \vec{r})$



density of states $N(E, \vec{r}) = N(E)$

Thus, the shifted density of states $N(E, \vec{r})$ can be related to the unshifted density of states $N(E, \vec{r}) = N(E)$. This corresponds to a substitution of the energy variable according to $E' = E + e\phi(\vec{r})$:

$$n_F(\vec{r}) \stackrel{(10.8)}{\underset{\varepsilon' = \varepsilon + e\phi(\vec{r})}{=}} \int_0^{E_F + e\phi(\vec{r})} d\varepsilon' N(\varepsilon', \vec{r}) = \int_0^{E_F} d\varepsilon' N(\varepsilon') + \int_{E_F}^{E_F + e\phi(\vec{r})} d\varepsilon' N(\varepsilon') \quad (10.9)$$

The first term coincides with the density n_0 of electrons in the absence of any potential. And for the second term it is reasonable to assume that the energy shift $e\phi(\vec{r})$ is small with respect to the Fermi energy E_F . In that case it is justified to consider the density of states in the interval $[E_F, E_F + e\phi(\vec{r})]$ to be approximately constant: $N(\varepsilon', \vec{r}) \approx N(E_F, \vec{r}) = N(E_F)$. Thus, (10.9) reduces to

$$n_F(\vec{r}) = n_0 + e\phi(\vec{r}) N(E_F) \quad (10.10)$$

This means that a local electrostatic potential $\phi(\vec{r})$ yields a local change of the electron density $n_F(\vec{r})$ in the Fermi sea.

10.2.4 Self-consistent Electrostatic Potential:

Now we embark upon a self-consistent calculation of the electrostatic potential $\phi(\vec{r})$. To this end we start with the total electron density, which amounts to the electron density of the Fermi sea described by (10.10) and two additional electrons outside of the Fermi sea:

$$n(\vec{r}) = \underbrace{n_0 + e\phi(\vec{r}) N(E_F)}_{\text{electrons in Fermi sea}} + \underbrace{\delta(\vec{r} - \vec{r}_1) + \delta(\vec{r} - \vec{r}_2)}_{\text{two electrons outside Fermi sea}} \quad (10.11)$$

Taking into account the total electron density (10.11) and the background charge density (10.5) yields finally the total charge density

$$\rho(\vec{r}) = \rho_I - en(\vec{r}) \quad (10.12)$$

Inserting (10.5) and (10.11) into (10.12) we observe that both contributions involving the density of electrons n_0 in the absence of any potential cancel each other:

$$\rho(\vec{r}) = -e \{ e\phi(\vec{r}) N(E_F) + \delta(\vec{r} - \vec{r}_1) + \delta(\vec{r} - \vec{r}_2) \} \quad (10.13)$$

Thus, we have derived with (10.13) that any electrostatic potential $\phi(\vec{r})$ induces a charge density $\rho(\vec{r})$. But, conversely, the Poisson equation describes that a charge density $\rho(\vec{r})$ leads to an electrostatic potential $\phi(\vec{r})$:

$$\Delta \phi(\vec{r}) = -\frac{1}{\varepsilon_0} \rho(\vec{r}) \quad (10.14)$$

Inserting (10.13) into (10.14) then yields a resulting differential equation, which determines the electrostatic po-

tential $\phi(\vec{r})$ self-consistently:

$$\left[\Delta - \frac{e^2 N(E_F)}{\epsilon_0} \right] \phi(\vec{r}) = \frac{e}{\epsilon_0} \{ \delta(\vec{r} - \vec{r}_1) + \delta(\vec{r} - \vec{r}_2) \} \quad (10.15)$$

Thus, the electrostatic potential $\phi(\vec{r})$ is determined by a sum of two Green functions of the Helmholtz equation:

$$(\Delta - \chi^2) G(\vec{r} - \vec{r}') = -\frac{e}{\epsilon_0} \delta(\vec{r} - \vec{r}') \quad (10.16)$$

Here we have introduced the Thomas-Fermi wave number

$$\chi^2 = \frac{e^2 N(E_F)}{\epsilon_0} \quad (10.17)$$

as an abbreviation.

10.2.5 Helmholtz Equation:

Performing a Fourier transformation of the Green function

$$G(\vec{r} - \vec{r}') = \int \frac{d^3 k}{(2\pi)^3} G(\vec{k}) e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} \quad (10.18)$$

converts the differential equation (10.16) to the algebraic equation

$$(-k^2 - \chi^2) G(\vec{k}) = -\frac{e}{\epsilon_0} \quad (10.19)$$

from which we can read off the Fourier transformed of the Green function:

$$G(\vec{k}) = \frac{e}{\epsilon_0} \frac{1}{k^2 + \chi^2} \quad (10.20)$$

Inserting (10.20) into (10.18) the Green function follows from evaluating the integral

$$G(\vec{r} - \vec{r}') = \frac{e}{\epsilon_0} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2 + \chi^2} e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} \quad (10.21)$$

We proceed by employing the Schwinger trick, which is a quite common and useful method within quantum field theory. To this end we start with the integral representation of the gamma function

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t} \quad (10.22)$$

With the substitution $t = a\tau$ we conclude

$$\frac{1}{a^z} = \frac{1}{\Gamma(z)} \int_0^\infty d\tau \tau^{z-1} e^{-a\tau} \quad (10.23)$$

In the particular case (10.21) we need (10.23) for $z=1$, which yields the elementary identity

$$\frac{1}{a} = \int_0^\infty d\tau e^{-a\tau} \quad (10.24)$$

Introducing in (10.21) the artificial Schwinger integral

(10.24) has the consequence that the wave vector integrals become Gaussian integrals

$$G(\vec{r}-\vec{r}') = \frac{e}{\epsilon_0} \int_0^\infty d\tau e^{-\lambda^2 \tau} \int \frac{d^3 k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{r} + i\vec{k}\cdot(\vec{r}-\vec{r}')} \quad (10.25)$$

which can be directly evaluated

$$G(\vec{r}-\vec{r}') = \frac{e}{8\pi^{3/2}\epsilon_0} \int_0^\infty d\tau \tau^{-\frac{3}{2}} e^{-\lambda^2 \tau - \frac{(\vec{r}-\vec{r}')^2}{4\tau}} \quad (10.26)$$

The remaining Schwinger integral (10.26) is of the type (3.471.9) in Gradshteyn

$$\int_0^\infty dx x^{\nu-1} e^{-\frac{\beta}{x} - \delta x} = \lambda \left(\frac{\beta}{\delta}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\delta}) \quad (10.27)$$

Thus, identifying $\nu = -\frac{1}{2}$, $\beta = \frac{(\vec{r}-\vec{r}')^2}{4}$, $\delta = \lambda^2$ converts (10.26) with the help of (10.27) to

$$G(\vec{r}-\vec{r}') = \frac{e}{4\pi^{3/2}\epsilon_0} \sqrt{\frac{2\lambda}{|\vec{r}-\vec{r}'|}} K_{-\frac{1}{2}}(|\vec{r}-\vec{r}'|\lambda) \quad (10.28)$$

According to (8.469.3) in Gradshteyn the modified Bessel function in (10.28) can be identified with an elementary function:

$$K_{\pm \frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \quad (10.29)$$

so (10.28) reduces to the Yukawa potential

$$G(\vec{r}-\vec{r}') = \frac{e}{4\pi\epsilon_0} \frac{e^{-|\vec{r}-\vec{r}'|\lambda}}{|\vec{r}-\vec{r}'|} \quad (10.30)$$

Note that in the formal limit $\lambda \rightarrow 0$ the Yukawa potential (10.30) reduces to the Coulomb potential as one would expect according to the Poisson equation emerging from the Helmholtz equation (10.16).

In conclusion, we read off from (10.16) and (10.30) the solution of the differential equation (10.15) for the electrostatic potential $\phi(\vec{r})$:

$$\phi(\vec{r}) = -\frac{e}{4\pi\epsilon_0} \left\{ \frac{e^{-|\vec{r}-\vec{r}'|\lambda}}{|\vec{r}-\vec{r}'|} + \frac{e^{-|\vec{r}-\vec{r}'|\lambda}}{|\vec{r}-\vec{r}'|} \right\} \quad (10.31)$$

Once the electrostatic potential $\phi(\vec{r})$ is determined self-consistently, we can deduce physical consequences from it. For instance, we show in the following how to derive from (10.31) the corresponding interaction potential between the two electrons.

10.2.6 Local Energy of Fermi Sea:

To this end we start with the local energy of the Fermi sea, which follows from multiplying the local density of states $N(\epsilon, \vec{r})$ with the energy ϵ and from an integration within the interval $[-e\phi(\vec{r}), E_F]$:

$$W_F(\vec{r}) = \int_{-e\phi(\vec{r})}^{E_F} d\epsilon \epsilon N(\epsilon, \vec{r}) \quad (10.32)$$

Similar to subsection 10.2.3 we proceed by reducing the shifted density of states $N(\epsilon, \vec{r})$ to the unshifted density of states $N(\epsilon, \vec{r}) = N(\epsilon)$ via the substitution $\epsilon' = \epsilon + e\phi(\vec{r})$:

$$W_F(\vec{r}) = \int_0^{E_F + e\phi(\vec{r})} d\epsilon' [\epsilon' - e\phi(\vec{r})] N(\epsilon') = \int_0^{E_F + e\phi(\vec{r})} d\epsilon' \epsilon' N(\epsilon') \\ + \int_{E_F}^{E_F + e\phi(\vec{r})} d\epsilon' \epsilon' N(\epsilon') - e\phi(\vec{r}) \int_0^{E_F + e\phi(\vec{r})} d\epsilon' N(\epsilon') \quad (10.33)$$

The respective terms are treated as follows. The first term corresponds to the energy W_0 in the absence of any potential $\phi(\vec{r})$. In the second term we can assume again that the integrand $\epsilon' N(\epsilon')$ is only slightly changing in the interval $[E_F, E_F + e\phi(\vec{r})]$, so the integrand is approximated by $E_F N(E_F)$. And the third term can be further evaluated by taking into account (10.9):

$$W_F(\vec{r}) = W_0 + E_F N(E_F) e\phi(\vec{r}) - e\phi(\vec{r}) n_F(\vec{r}) \quad (10.34)$$

Applying (10.10) in the second term yields

$$W_F(\vec{r}) = W_0 + E_F [n_F(\vec{r}) - n_0] - e\phi(\vec{r}) n_F(\vec{r}) \quad (10.35)$$

Furthermore, combining (10.10) and (10.11) allows to express the local energy of the Fermi sea in terms of the electrostatic potential $\phi(\vec{r})$ and the total electron density $n(\vec{r})$:

$$W_F(\vec{r}) = W_0 + [E_F - e\phi(\vec{r})] [n(\vec{r}) - \delta(\vec{r} - \vec{r}_1) - \delta(\vec{r} - \vec{r}_2)] - E_F n_0 \quad (10.36)$$

10.2.7 Local Total Energy:

In order to obtain the local total energy we have to take into account apart from the local energy of the Fermi sea two other energy contributions:

- 1) the local energy of the positive background charge, which amounts to

$$W_I(\vec{r}) = e n_0 \phi(\vec{r}) \quad (10.37)$$

- 2) the local energy of the two electrons outside of the Fermi sea

$$W_{ze}(\vec{r}) = [E_F - e\phi(\vec{r})] [\delta(\vec{r} - \vec{r}_1) + \delta(\vec{r} - \vec{r}_2)] \quad (10.38)$$

Adding all three contributions

$$W(\vec{r}) = W_F(\vec{r}) + W_I(\vec{r}) + W_{ze}(\vec{r}) \quad (10.39)$$

leads with (10.36) - (10.38) the result that the contributions of all delta functions cancel:

$$W(\vec{r}) = W_0 + [E_F - e\phi(\vec{r})] [n(\vec{r}) - n_0] \quad (10.40)$$

10.2.8 Interaction Potential:

Integrating the difference $W(\vec{r}) - W_0$ over space yields the energy, which emerges due to the presence of two electrons outside of the Fermi sea. For combinatorial reasons we have to divide this energy by two in order to obtain the total interaction energy:

$$V_{tot}(\vec{r}_1, \vec{r}_2) = \frac{1}{2} \int d^3r [W(\vec{r}) - W_0] \quad (10.41)$$

Inserting (10.40) in (10.41) leads to two terms

$$V_{tot}(\vec{r}_1, \vec{r}_2) = \frac{1}{2} E_F \int d^3r [n(\vec{r}) - n_0] - \frac{1}{2} \int d^3r e\phi(\vec{r}) [n(\vec{r}) - n_0] \quad (10.42)$$

As the number of electrons is conserved, we have to demand

$$\int d^3r [n(\vec{r}) - n_0] = 0 \quad (10.43)$$

so (10.42) reduces to

$$V_{tot}(\vec{r}_1, \vec{r}_2) = -\frac{e}{2} \int d^3r \phi(\vec{r}) [n(\vec{r}) - n_0] \quad (10.44)$$

Taking into account (10.11), the total energy (10.44) can finally only be expressed in terms of the electrostatic potential $\phi(\vec{r})$:

$$V_{tot}(\vec{r}_1, \vec{r}_2) = -\frac{e}{2} \int d^3r \phi(\vec{r}) [eN(E_F)\phi(\vec{r}) + \delta(\vec{r} - \vec{r}_1) + \delta(\vec{r} - \vec{r}_2)] \quad (10.45)$$

Here we have to step back and remind us about the underlying philosophy of the whole calculation. Basically, we investigate how the additional presence of two electrons changes the properties of the Fermi sea. To this end we considered the electrostatic potential $\phi(\vec{r})$ to be a small perturbation in comparison with the Fermi energy E_F , see e.g. (10.10) and its derivation. From this point of view we recognise in (10.45) two different contributions, the first (second) one being linear (quadratic) in the electrostatic potential $\phi(\vec{r})$. As we restrict ourselves to a linear response calculation, we can consider in (10.45) only the first-order term in $\phi(\vec{r})$ as the

interaction energy between the two electrons. If we were interested in higher-order contributions in the electrostatic potential $\phi(\vec{r})$, we would have to take them into account already at the beginning of the derivation, i. e. at approximating (10.9) by (10.10). With this consideration at hand we obtain from (10.45)

$$V_{\text{tot}}(\vec{r}_1, \vec{r}_2) = -\frac{e}{2} [\phi(\vec{r}_1) + \phi(\vec{r}_2)] \quad (10.46)$$

Inserting (10.31) in (10.46) yields infinitely large self-energies, which we ignore due to an appropriate renormalization of the energy scale, so we obtain

$$V(\vec{r}_1, \vec{r}_2) = \frac{e^2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_1 - \vec{r}_2|} e^{-|\vec{r}_1 - \vec{r}_2|/\lambda} \quad (10.47)$$

Thus, the interaction potential between two electrons in the presence of a Fermi sea is of a Yukawa-type. It is exponentially suppressed beyond the screening length λ , which is defined by the inverse of the Thomas-Fermi wave number. This length scale is of the order of 10 nm in typical metals. While we, thus, find a strong suppression of the repulsive interaction at large distances, there is no sign of it becoming negative.

10.3 Virtual Phonon Exchange:

In the previous section we have seen that the Fermi sea only screens the Coulomb repulsion between two electrons. In order to obtain a residual attractive interaction between two electrons, one needs a virtual exchange of quasi-particles. In this section we treat the concrete example of a virtual exchange of phonons between the two electrons of a Cooper pair as was suggested for the first time by Frohlich:

electron 1 $\xleftrightarrow{\text{red}} \text{virtual phonons} \xleftrightarrow{\text{blue}}$ electron 2 (10.48)

10.3.1 Basic Ideas:

A phonon is a quantized, collective motion of all ion cores. In order to describe the virtual exchange of phonons properly one would have to consider the whole phonon spectrum. For the sake of simplicity, however, we restrict ourselves in this section to the elementary process of a virtual exchange of a single phonon between the two electrons of a Cooper pair. Subsequently, the next chapter deals with the con-

responding many-body problem. Thus, the basic ideas here are the following:

1) We assume that two electrons with antiparallel wave vectors $(\vec{k}, -\vec{k})$, bound in a Cooper pair, have an energetically lower state than two single electrons with wave vectors (\vec{k}_1, \vec{k}_2) . That this assumption is justified was discussed in the last chapter with the Cooper problem.

2) Having before and after the virtual phonon exchange a Cooper pair with wave vectors $(\vec{k}, -\vec{k})$ and $(\vec{k}', -\vec{k}')$, the total momentum is conserved by this exchange process

$$\vec{k} = \vec{k} + (-\vec{k}) = \vec{k}' + (-\vec{k}') = \vec{0} \quad (10.49)$$

These considerations allow for a rough comparison between a normal conductor and a superconductor. In a normal conductor the virtual exchange of phonons between electrons does not fulfill the conservation of momentum (10.49) and, as a consequence, a resistive Ohmic resistance occurs. In contrast to that in a superconductor the total momentum is conserved by a virtual exchange of phonons and electrons in a Cooper pair as is summarised in (10.49). This explains qualitatively why no Ohmic resistance occurs for a superconductor.

10.3.2 Notation:

Now we focus on describing such virtual phonon exchange processes between the two electrons of a Cooper pair. To this end we introduce the following notation:

- 1) The first (second) electron in the Cooper pair is denoted by ① (②).
- 2) The state before (after) the phonon exchange is characterised by I (II).
- 3) The energy of an electron (with respect to the Fermi energy E_F) with wave vector \vec{k} is abbreviated by $E_{\vec{k}}$ ($E_{\vec{k}}$).
- 4) Correspondingly, the energy before (after) the phonon exchange with respect to the Fermi energy E_F reads E_I (E_{II}).

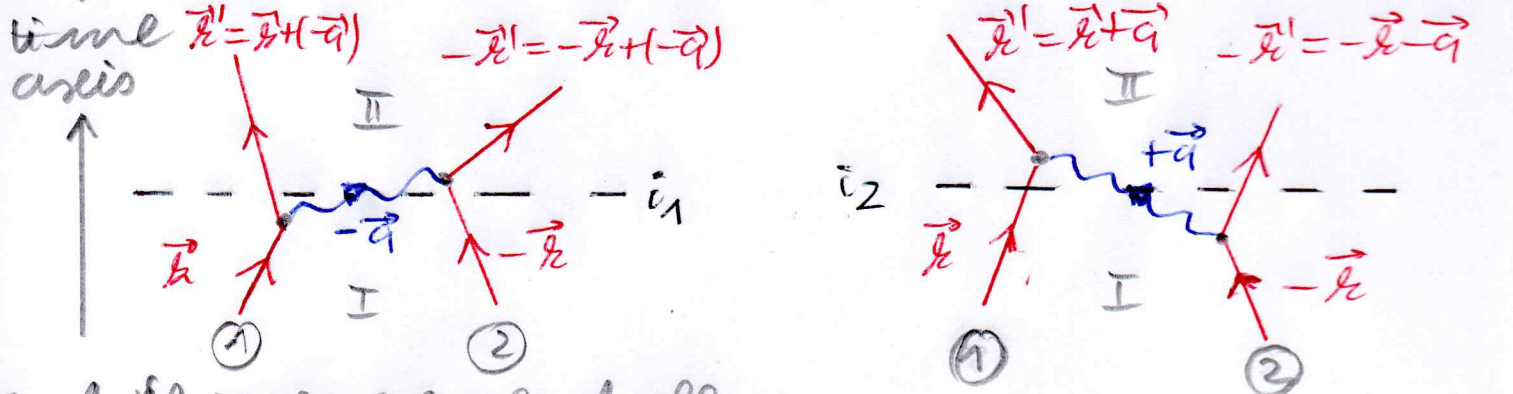
- 5) The intermediate state, where one electron has just emitted a phonon and the other electron has not yet absorbed it, is abbreviated by i .
- 6) The phonon dispersion, whose concrete shape will be discussed in the next chapter, is denoted by $\omega_{\vec{q}}$.

10.3.3 Elementary Processes:

Assuming that the considered material is reflection-symmetric, there are two symmetric elementary processes, which fulfill the assumptions listed in subsection 10.3.1. Due to symmetry reasons, this reflection symmetry of the material in real space leads to a corresponding reflection symmetry in Fermi space. Therefore, the dispersion relations of both electrons and phonons are assumed to be reflection-symmetric:

electrons: $\epsilon_{\vec{k}} = \epsilon_{-\vec{k}}$, phonons: $\omega_{\vec{q}} = \omega_{-\vec{q}}$ (10.50)

The two possible symmetric processes with regard to a virtual exchange of a phonon are then depicted by Feynman diagrams as follows:



In both cases we read off

$E_I = \epsilon_{\vec{k}} + \epsilon_{-\vec{k}} \stackrel{(10.50)}{=} 2\epsilon_{\vec{k}} \quad (10.51)$

$E_{II} = \epsilon_{\vec{k}'} + \epsilon_{-\vec{k}'} \stackrel{(10.50)}{=} 2\epsilon_{\vec{k}'} \quad (10.52)$

The respective intermediate states are characterized by

	i_1	i_2
electron 1	$\vec{k}' = \vec{k} + (-\vec{q})$	\vec{k}
electron 2	$-\vec{k}$	$-\vec{k}' = -\vec{k} - \vec{q}$
phonon	$-\vec{q}$	\vec{q}
i	$E_{i_1} = \epsilon_{\vec{k}'} + \epsilon_{-\vec{k}} + \omega_{-\vec{q}} \quad (10.53)$	$E_{i_2} = \epsilon_{-\vec{k}'} + \epsilon_{\vec{k}} + \omega_{\vec{q}} \quad (10.54)$

due to the assumed reflection symmetry of the material, we read off from (10.50), (10.53) and (10.54)

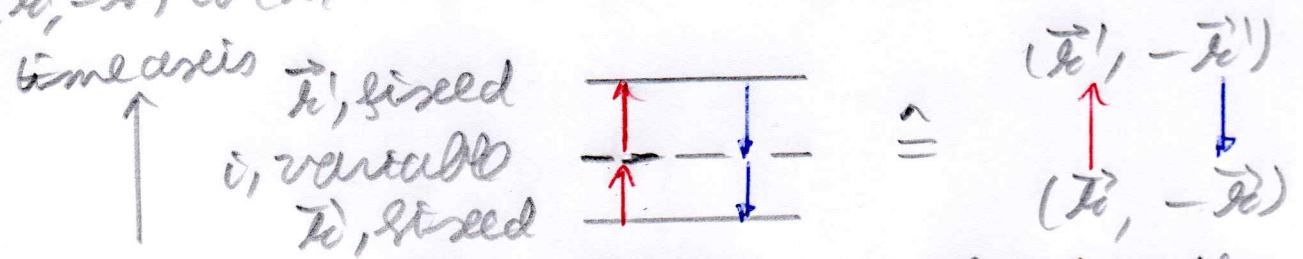
$$E_i = E_{i1} = E_{i2} = E_{\vec{k}1} + E_{\vec{k}} + \frac{1}{2} \omega_{\vec{q}} \quad (10.55)$$

Note that this equality of the energy of the two intermediate states reflects the symmetry of both processes.

10.3.4 Perturbation Theory:

Now we investigate the virtual phonon exchange between the two electrons of a Cooper pair within perturbation theory. It turns out that first-order perturbation theory just yields a shift of the unperturbed energy by a constant amount of energy. Thus, this first-order perturbative energy correction can be renormalised by a corresponding shift of the zero-point of the energy scale. Only second-order perturbation theory yields a first physically relevant energy correction.

The cohesion ("Zusammenhalt") of the two electrons in a Cooper pair finally emerges from the virtual exchange of phonons. Therefore, it is necessary to consider not only the scattering process from $(\vec{k}, -\vec{k})$ to $(\vec{k}', -\vec{k}')$ but also the inverse scattering process from $(\vec{k}', -\vec{k}')$ to $(\vec{k}, -\vec{k})$ within perturbation theory.



Here we highlight in red (—) and blue (—) the first and second process occurring in positive and negative time direction, respectively:

$$V_{\vec{k}, \vec{k}'}^{(1)} = \sum_{i1} \frac{\langle \Pi | V_{e-p} | i1 \rangle \langle i1 | V_{e-p} | I \rangle}{E_I - E_{i1}} \quad (10.56)$$

$$V_{\vec{k}, \vec{k}'}^{(2)} = \sum_{i2} \frac{\langle I | V_{e-p} | i2 \rangle \langle i2 | V_{e-p} | \Pi \rangle}{E_{i2} - E_I} \quad (10.57)$$

With this we obtain a dynamic picture that the Cooper pairs $(\vec{k}, -\vec{k})$ and $(\vec{k}', -\vec{k}')$ continuously exchange phonons between each other. In order to get the resulting matrix element $V_{\vec{k}, \vec{k}'}$ it only remains to con-

sider the average

$$V_{\vec{k}, \vec{k}'} = \frac{1}{2} (V_{\vec{k}, \vec{k}'}^{(1)} + V_{\vec{k}, \vec{k}'}^{(2)}) \quad (10.58)$$

From (10.56) - (10.58) we yield

$$V_{\vec{k}, \vec{k}'} = \frac{1}{2} \left\{ \sum_{i_1} \frac{\langle \text{II} | V_{e-p} | i_1 \rangle \langle i_1 | V_{e-p} | \text{I} \rangle}{E_{\text{I}} - E_{i_1}} + \sum_{i_2} \frac{\langle \text{I} | V_{e-p} | i_2 \rangle \langle i_2 | V_{e-p} | \text{II} \rangle}{E_{\text{II}} - E_{i_2}} \right\} \quad (10.59)$$

We remark that (10.59) can be simplified in two respects:

1) The matrix elements are Hermitian

$$\langle \text{II} | V_{e-p} | i \rangle \langle i | V_{e-p} | \text{I} \rangle = \langle \text{I} | V_{e-p} | i \rangle \langle i | V_{e-p} | \text{II} \rangle \quad (10.60)$$

2) The energy differences in the denominators of (10.59) read in detail

$$E_{\text{I}} - E_{i_1} \stackrel{(10.51), (10.55)}{=} E_{\vec{k}} - E_{\vec{k}'} - \hbar \omega_{\vec{q}} \quad (10.61)$$

$$E_{\text{II}} - E_{i_2} \stackrel{(10.52), (10.55)}{=} E_{\vec{k}'} - E_{\vec{k}} - \hbar \omega_{\vec{q}} \quad (10.62)$$

Thus, introducing as a new abbreviation the energy difference

$$\hbar \omega_{\vec{k}, \vec{k}'} = E_{\vec{k}'} - E_{\vec{k}} \quad (10.63)$$

(10.61), (10.62) reduce to

$$E_{\text{I}} - E_{i_1} = -\hbar (\omega_{\vec{k}, \vec{k}'} + \omega_{\vec{q}}) \quad (10.64)$$

$$E_{\text{II}} - E_{i_2} = \hbar (\omega_{\vec{k}, \vec{k}'} - \omega_{\vec{q}}) \quad (10.65)$$

respectively.

Thus, inserting (10.60) and (10.64), (10.65) in (10.59) yields for the matrix element of a virtual phonon exchange

$$\begin{aligned} V_{\vec{k}, \vec{k}'} &= \sum_i \frac{\langle \text{II} | V_{e-p} | i \rangle \langle i | V_{e-p} | \text{I} \rangle}{\hbar} \cdot \frac{1}{2} \left(\frac{-1}{\omega_{\vec{k}, \vec{k}'} + \omega_{\vec{q}}} + \frac{1}{\omega_{\vec{k}, \vec{k}'} - \omega_{\vec{q}}} \right) \\ &= \sum_i \frac{\langle \text{II} | V_{e-p} | i \rangle \langle i | V_{e-p} | \text{I} \rangle}{\hbar} \frac{\omega_{\vec{q}}}{\omega_{\vec{k}, \vec{k}'}^2 - \omega_{\vec{q}}^2} \quad (10.66) \end{aligned}$$

Note that the energy $\hbar \omega_{\vec{q}}$ of the phonon being exchanged between the two electrons of a Cooper pair can be larger or smaller than the energy difference $\hbar \omega_{\vec{k}, \vec{k}'}$ between the final and the initial state of a single electron in a Cooper pair. Thus, the matrix element (10.66) can be either negative, i.e. attractive, or positive, i.e. repulsive.