4. London Equations:

In Chapters 1 and 2 we discussed that the seminal experiments of Onnes in 1911 as well as of Meissner and Ochsenfeld in 1933 were decisive for characterising superconductors as representing at the same time ideal conductors and ideal diamagnets. The brothers Fritz and Heinz London achieved in 1935 to explain these experimentally found electric and magnetic properties of superconductors within a phenomenological theory.

4.1 London Equations:

Starting point of the London theory is the assumption, borrowed from Landau, that the superconductor contains two different sorts of electrons, namely the normal conducting and the superconducting electrons, which are distinguished by the index "n" and "s". The respective current densities $\vec{j}_n$ and $\vec{j}_n$ can not be measured separately in an experiment, but they contribute both together to the total current density

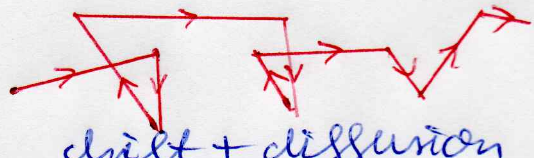
$$\vec{j} = \vec{j}_n + \vec{j}_s \tag{4.1}$$

We have now to discuss in more detail how normal and superconducting electrons are described.

4.1.1 Normal Conducting Electrons:

In a metal normal conducting electrons scatter successively at the atomic cores, which leads to permanent changes of the direction of motion. Therefore, normal conducting electrons are exposed to a resistance.

More precisely, the motion of a normal conducting electron is affected by two causes.



drift + diffusion

On the one hand, an electron performs a thermal motion due to the finite temperature of the metal, which is also called diffusion. On the thermal average, the position, the velocity, and the acceleration vanish:

$$\overline{\vec{s}} = \vec{0}\,, \quad \overline{\vec{v}} = \vec{0}\,, \quad \overline{\vec{a}} = \vec{0} \tag{4.2}$$

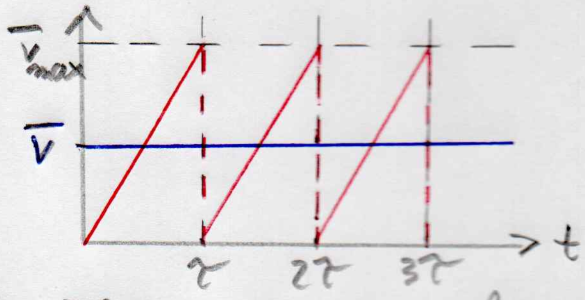
On the other hand, an external constant electric field yields a drift motion, which is described within the Drude model as follows. During the mean free time τ the normal conducting electron is constantly accelerated with

$$m\vec{a} = e\vec{E} \quad \Rightarrow \quad \vec{a} = \frac{e}{m}\vec{E} \tag{4.3}$$

and acquires after τ the velocity

$$\vec{v}_{max} = \vec{a}\,\tau \overset{(4.3)}{=} \frac{e}{m}\vec{E}\,\tau \tag{4.4}$$

After the mean free time τ the normal conducting electron scatters with an atomic core and, thus, gets suddenly decelerated to a vanishing velocity. Here the scattering time is negligible in comparison with the mean time τ.



The successive changes from acceleration during mean free time τ and sudden deceleration yields approximately an averaged velocity:

$$\vec{\bar{v}} = \frac{1}{2}\vec{v}_{max} \overset{(4.4)}{=} \frac{1}{2}\frac{e}{m}\vec{E}\,\tau \tag{4.5}$$

Taking into account the density of normal conducting electrons n_n, this yields the averaged normal conducting current density

$$\vec{j}_n = e\,n_n\,\vec{\bar{v}} \overset{(4.5)}{=} \frac{e^2 n_n \tau}{2m}\vec{E} \tag{4.6}$$

A comparison with the Ohm law

$$\vec{j}_n = \sigma_n \vec{E} \tag{4.7}$$

allows to read off then the electric conductivity of the normal conducting electrons:

$$\sigma_n = \frac{e^2 n_n \tau}{2m} \tag{4.8}$$

Note that here e and m denote charge and mass of the normal conducting electrons.

4.1.2 Superconducting Electrons:

Here we assume that the superconducting electrons move without scatterings and without any electric resistance. In view of their microscopic realisations within the BCS theory in terms of Cooper pairs we introduce for the superconducting electrons the charge e_s and the mass m_s, where we specify later on

$$e_s = 2e, \quad m_s = 2m \tag{4.9}$$

The scattering-free motion of superconducting electrons is determined by the Newton law

$$m_s \frac{d\vec{v}_s}{dt} = e_s \vec{E} \tag{4.10}$$

yielding the superconducting current density

$$\vec{j}_s = e_s n_s \vec{v}_s \tag{4.11}$$

Here n_s denotes the density of superconducting electrons. As the velocity $\vec{v}_s$ is not directly experimentally accessible, we eliminate it from (4.10) and (4.11)

$$\frac{d}{dt}\left(\frac{m_s}{e_s^2 n_s}\vec{j}_s\right) = \vec{E} \tag{4.12}$$

As is used in fluid dynamics, a total time derivative of a field, which depends on both space and time, yields according to the chain rule two contributions:

$$\frac{d}{dt} \bullet = \frac{\partial}{\partial t} \bullet + (\vec{v}_s \cdot \vec{\nabla}) \bullet \tag{4.13}$$

The first term is a spatially local time derivative, which stems from a temporal change of the quantity at one space point, whereas the second term represents a transport derivative, which is caused by the particle motion. From (4.12) and (4.13) we conclude

$$\left[\frac{\partial}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\right]\left(\frac{m_s}{e_s^2 n_s}\vec{j}_s\right) = \vec{E} \tag{4.14}$$

For a given electric field strength $\vec{E}$, (4.14) allows to determine the corresponding superconducting current density $\vec{j}_s$. But provided that $\vec{E}$ does not have any spatial dependence or that it can be neglected, this turns out to be the case also for $\vec{j}_s$. In that case (4.14) reduces to the first London equation

$$\frac{\partial}{\partial t}\left(\Lambda_s \vec{j}_s\right) = \vec{E} \tag{4.15}$$

with the abbreviation

$$\Lambda_s = \frac{m_s}{e_s^2 n_s} \tag{4.16}$$

4.1.3 Remarks:

From the above follows that normal and superconducting electrons behave differently. In case that the electric field strength $\vec{E}$ vanishes, we conclude:

1) The Ohm law (4.7) implies for $\vec{E}=\vec{0}$ that the normal conducting current density vanishes:
$$\vec{E}=\vec{0} \xrightarrow{(4.7)} \vec{j}_n=\vec{0} \qquad (4.17)$$

2) The first London equation (4.15) has for $\vec{E}=\vec{0}$ the consequence to yield a constant superconducting current density:
$$\vec{j}_s = \text{const.} \qquad (4.18)$$

4.1.4 Induction Law:

The differential form of the induction law reads according to Maxwell in SI units
$$\operatorname{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (4.19)$$
Applying the integral theorem of Stokes converts this differential form into its corresponding integral form
$$\int_F \operatorname{rot}\vec{E}\cdot d\vec{F} = \oint_{\partial F} \vec{E}\cdot d\vec{r} = -\int_F \frac{\partial \vec{B}}{\partial t}\cdot d\vec{F} \qquad (4.20)$$
Thus, provided that the area F is not changing with time
$$\text{"}\frac{\partial}{\partial t}F = 0\text{"} \qquad (4.21)$$
we obtain that the induction voltage along the closed curve ∂F
$$U_{ind} = \oint_{\partial F} \vec{E}\cdot d\vec{r} \qquad (4.22)$$
is given by the temporal derivative of the magnetic flux through the area F, which has ∂F as its boundary
$$\Phi = \int_F \vec{B}\cdot d\vec{F} \qquad (4.23)$$
The additional minus sign in the resulting integral form of the induction law
$$U_{ind} = -\frac{\partial \Phi}{\partial t} \qquad (4.24)$$
represents the Lenz rule. Any induction effect is always acting against its origin.

4.1.5 Second London Equation:

Applying the rotation to the first London equation (4.15) and interchanging temporal as well as spatial derivatives yields

$$\frac{\partial}{\partial t} \operatorname{rot}(\Lambda_s \vec{j}_s) = \operatorname{rot} \vec{E} \qquad (4.25)$$

Inserting the differential form of the induction law (4.19), we conclude

$$\frac{\partial}{\partial t} \{\operatorname{rot}(\Lambda_s \vec{j}_s) + \vec{B}\} = \vec{0} \qquad (4.26)$$

Thus, we obtain the conserved quantity

$$\operatorname{rot}(\Lambda_s \vec{j}_s) + \vec{B} = \text{const.} \qquad (4.27)$$

In the volume of a superconductor the following relations hold:

1) Within the framework of the London theory the density of superconducting electrons n_s and, thus, the abbreviation Λ_s in (4.16) are spatially constant. This restriction is loosened in the Ginzburg-Landau theory.
2) From the first London equation (4.15) we concluded for a vanishing electric field $\vec{E} = \vec{0}$ that the superconducting current density $\vec{j}_s$ is constant according to (4.18).
3) The Meissner-Ochsenfeld effect implies

$$\vec{B} = \vec{0} \qquad (4.28)$$

Thus, the constant at the right-hand side of (4.27) must vanish and we obtain the second London equation

$$\operatorname{rot}(\Lambda_s \vec{j}_s) + \vec{B} = \vec{0} \qquad (4.29)$$

4.1.6 Conclusions:

From (4.29) and the Helmholtz vector decomposition theorem we conclude that the magnetic induction $\vec{B}$ can be represented as the rotation of a vector potential $\vec{A}$:

$$\vec{B} = \operatorname{rot} \vec{A} \qquad (4.30)$$

Inserting (4.30) into (4.29) yields

$$\operatorname{rot}(\Lambda_s \vec{j}_s + \vec{A}) = \vec{0} \qquad (4.31)$$

Thus, apart from the gauge freedom of an additional gradient field, the vector potential is uniquely determined by

$$\vec{A} = -\Lambda_s \vec{j}_s \qquad (4.32)$$

Inserting (4.16) in (4.32) we then obtain for the superconducting current density

$$\vec{j}_s = -\frac{e_s^2 n_s}{m_s} \vec{A} \qquad (4.33)$$

Comparing (4.11) with (4.33) we finally identify the velocity of superconducting electrons with

$$\vec{v}_s = -\frac{e_s}{m_s} \vec{A} \qquad (4.34)$$

4.2 Field Equations for a Superconducting Medium

In order to describe the matter and the field state in a superconductor, we use the following quantities. The matter state is described by the charge density ϱ and the current density $\vec{j}$, whereas the field state is described by the electric field strength $\vec{E}$ and the magnetic induction $\vec{B}$. Note that all fields are considered as quantities of space and time.

4.2.1 System of Equations:

The Maxwell theory consists of four field equations

$$\operatorname{div} \vec{E} = \frac{\varrho}{\varepsilon_0} \quad (M1) \qquad \operatorname{rot} \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \quad (M2)$$

$$\operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (M3) \qquad \operatorname{div} \vec{B} = 0 \quad (M4)$$

with the identity

$$c^2 = \frac{1}{\varepsilon_0 \mu_0} \qquad (M5)$$

Thus, the basic idea is that a given $\varrho, \vec{j}$ yield an electromagnetic field $\vec{E}, \vec{B}$. Note that it is a valid approximation for a superconductor to assume for the relative dielectric constant ε_r and the relative permeability μ_r their respective vacuum values, i.e.

$$\varepsilon_r \approx 1, \quad \mu_r \approx 1 \qquad (M6)$$

In addition, the London theory is based on the idea of having both a normal and a superconducting contribution of the current density

$$\vec{j} = \vec{j}_n + \vec{j}_s \qquad (L3)$$

where the former obeys the Ohm law

$$\vec{j}_n = \sigma_n \vec{E} \qquad (L4)$$

and the latter fulfill the two London equations

$$\frac{\partial}{\partial t}(\Lambda_s \vec{j}_s) = \vec{E} \quad (L1) \qquad \mathrm{rot}(\Lambda_s \vec{j}_s) = -\vec{B} \quad (L2)$$

4.2.2 Consistency Relations:

At first we conclude from (M1) and (M2) the continuity equation for ρ, $\vec{j}$:

$$\frac{\partial}{\partial t}\rho \overset{(M1)}{=\!=} \varepsilon_0 \,\mathrm{div}\, \frac{\partial \vec{E}}{\partial t} \overset{(M2)}{=\!=} -\varepsilon_0 \mu_0 c^2 \,\mathrm{div}\, \vec{j} \overset{(M5)}{=\!=} -\mathrm{div}\, \vec{j} \quad (M7)$$

Furthermore, we observe that the rotation of the first London equation corresponds to the third Maxwell equation:

$$\mathrm{rot}\, \vec{E} \overset{(L1)}{=\!=} \frac{\partial}{\partial t} \mathrm{rot}(\Lambda_s \vec{j}_s) \overset{(L2)}{=\!=} -\frac{\partial \vec{B}}{\partial t} \;\hat{=}\; (M3)$$

And, correspondingly, the divergence of the second London equation yields the fourth Maxwell equation

$$\mathrm{div}\, \vec{B} \overset{(L2)}{=\!=} 0 \;\hat{=}\; (M4)$$

Thus, one can argue that the London theory substitutes (M3), (M4) by the more microscopic London equations (L1), (L2) for a superconductor.

4.2.3 Elimination of Current Densities:

The normal and superconducting components of the current density $\vec{j}_n$, $\vec{j}_s$ are not directly experimentally accessible. Therefore, we follow the strategy to eliminate them from the above system of equations. At first, we eliminate the normal conducting current density $\vec{j}_n$ from (L3) and (L4):

$$\vec{j}_s = \vec{j} - \sigma_n \vec{E} \tag{4.35}$$

Afterwards, we eliminate the superconducting current density $\vec{j}_s$ in the London equations (L1) and (L2). From (L1) and (4.35) we obtain a relation between $\vec{j}$ and $\vec{E}$:

$$\vec{E} + \frac{\partial}{\partial t}(\Lambda_s \sigma_n \vec{E}) = \frac{\partial}{\partial t}(\Lambda_s \vec{j}) \tag{4.36}$$

correspondingly, we deduce from (L2) and (4.35) an analogous relation between $\vec{j}$ and $\vec{B}$:

$$-\vec{B} = \mathrm{rot}(\Lambda_s \vec{j}) - \mathrm{rot}(\Lambda_s \sigma_n \vec{E})$$

$$\overset{(M3)}{\Longrightarrow} \vec{B} + \frac{\partial}{\partial t}(\Lambda_s \sigma_n \vec{B}) = -\mathrm{rot}(\Lambda_s \vec{j}) \tag{4.37}$$

4.2.4 Field Equations:

We now derive equations for the electric field $\vec{E}$ and the magnetic field $\vec{B}$ alone. In the former case we have to eliminate the right-hand side of (4.36). To this end we determine the time derivative of (M2):

$$\text{rot}\,\frac{\partial \vec{B}}{\partial t} = \mu_0 \frac{\partial \vec{j}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \tag{4.38}$$

Inserting (M3) thus yields

$$\mu_0 \frac{\partial \vec{j}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \text{rot}\,\text{rot}\,\vec{E} \tag{4.39}$$

Taking into account the vector identity

$$\text{rot}\,\text{rot} = \text{grad}\,\text{div} - \Delta \tag{4.40}$$

as well as (M1), we then obtain

$$\mu_0 \frac{\partial \vec{j}}{\partial t} = \Delta \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{1}{\varepsilon_0}\,\text{grad}\,\rho \tag{4.41}$$

Inserting (4.41) into (4.36) we finally get one differential equation determining the electric field

$$\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \Delta \vec{E} + \mu_0 \sigma_n \frac{\partial \vec{E}}{\partial t} + \frac{\mu_0}{\Lambda_S}\vec{E} = -\frac{1}{\varepsilon_0}\,\text{grad}\,\rho \tag{4.42}$$

In an analogous way we now proceed for the magnetic field and eliminate the right-hand side of (4.37). To this end we evaluate the rotation of (M2):

$$\text{rot}\,\text{rot}\,\vec{B} = \mu_0\,\text{rot}\,\vec{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\,\text{rot}\,\vec{E} \tag{4.43}$$

Inserting (M3), (M4) and the vector identity (4.40) yields

$$\mu_0\,\text{rot}\,\vec{j} = -\Delta \vec{B} + \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} \tag{4.44}$$

Thus, combining (4.36) and (4,44) we finally obtain

$$\frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} - \Delta \vec{B} + \mu_0 \sigma_n \frac{\partial \vec{B}}{\partial t} + \frac{\mu_0}{\Lambda_S}\vec{B} = \vec{0} \tag{4.45}$$

We conclude that both the electric and the magnetic field $\vec{E}$ and $\vec{B}$ in a superconductor fulfill an extended homogeneous and inhomogeneous telegraph equation, respectively. The additional term involves a new length scale λ_L, which is given by

$$\frac{1}{\lambda_L^2} = \frac{\mu_0}{\Lambda_S} \implies \lambda_L = \sqrt{\frac{\Lambda_S}{\mu_0}} \tag{4.46}$$

and reduces with (4.16) to

$$\lambda_L = \sqrt{\frac{m_S}{e_S^2 n_S \mu_0}} \tag{4.47}$$

In the following we explore the physical meaning of that new length scale.

4.3 Stationary case:

The stationary case of a superconductor is defined by neglecting all temporal derivatives. At first, we conclude from (M3)

$$\frac{\partial \vec{E}}{\partial t} = \vec{0} , \quad \text{rot}\, \vec{E} = \vec{0} \qquad (4.48)$$

Without loss of generality the electric field thus vanishes

$$\vec{E} = \vec{0} \qquad (4.49)$$

so that also the normal conducting current density $\vec{j}_n$ vanishes as was already concluded in (4.17). As a consequence we obtain from (4.42)

$$\frac{\partial \rho}{\partial t} = 0 , \quad \text{grad}\, \rho = \vec{0} \qquad (4.50)$$

so that also the charge density must vanish

$$\rho = 0 \qquad (4.51)$$

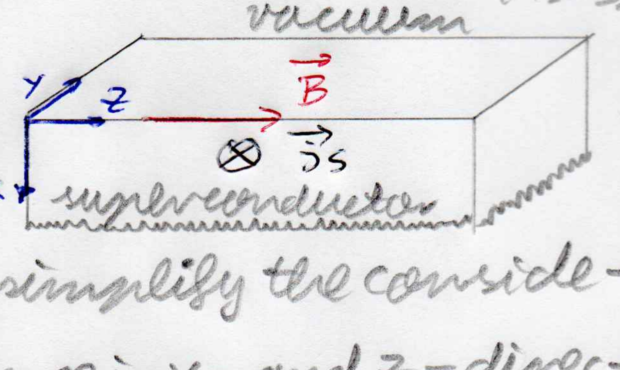
But for the magnetic field we conclude in the stationary case from (4.45) together with (4.46) that it solves the Helmholtz equation

$$\Delta \vec{B} - \frac{1}{\lambda_L^2} \vec{B} = \vec{0} \qquad (4.52)$$

And for the current density $\vec{j}$, which consists according to (L3) and (4.17) only of the superconducting current density $\vec{j}_s$, we obtain the same Helmholtz equation. Namely, the rotation of (L2) yields with (M2), (M7), (4.40), and (4.50)

$$\Delta \vec{j}_s - \frac{1}{\lambda_L^2} \vec{j}_s = \vec{0} \qquad (4.53)$$

Let us now investigate the consequences of the London theory for the geometry of a superconducting half space:



This special geometry allows to simplify the considerations as follows:

1) Due to translational invariance in y- and z-direction, both the magnetic field $\vec{B}$ and the superconducting current density $\vec{j}_s$ can only depend on x:

$$\vec{B} = \vec{B}(x) , \quad \vec{j}_s = \vec{j}_s(x) \qquad (4.54)$$

2) From (M2) and (4.17) we conclude:

$$\mathrm{rot}\,\vec{B} = \mu_0 \vec{j}_s \quad \xRightarrow{(4.54)} \quad \begin{pmatrix} 0 \\ -\frac{\partial B_z}{\partial x} \\ \frac{\partial B_y}{\partial x} \end{pmatrix} = \mu_0 \begin{pmatrix} j_{sx} \\ j_{sy} \\ j_{sz} \end{pmatrix} \qquad (4.55)$$

Thus, the x-component of both $\vec{B}$ and $\vec{j}_s$ vanish and the magnetic field as well as the superconducting current density lie in the YZ-plane.

3) As we still have the freedom to choose a particular coordinate system, we assume without loss of generality

$$\vec{B}(x) = B_z(x)\,\vec{e}_z \qquad (4.56)$$

From (4.52) and (4.56) we obtain for the solution in the superconductor:

$$\vec{B}(x) = B_z(0)\, e^{-x/\lambda_L}\,\vec{e}_z, \qquad x \geq 0 \qquad (4.57)$$

Assuming in the vacuum the magnetic field

$$\vec{B}(x) = \mu_0 \vec{H}(x), \quad \vec{H}(x) = H_0 \vec{e}_z, \quad x \leq 0 \qquad (4.58)$$

the continuity condition for the transversal component of the magnetic field strength $\vec{H}$ at the boundary imposes

$$B_z(0) = \mu_0 H_0 \qquad (4.59)$$

Furthermore, inserting (4.57), (4.59) in (4.55) yields the corresponding superconducting current density

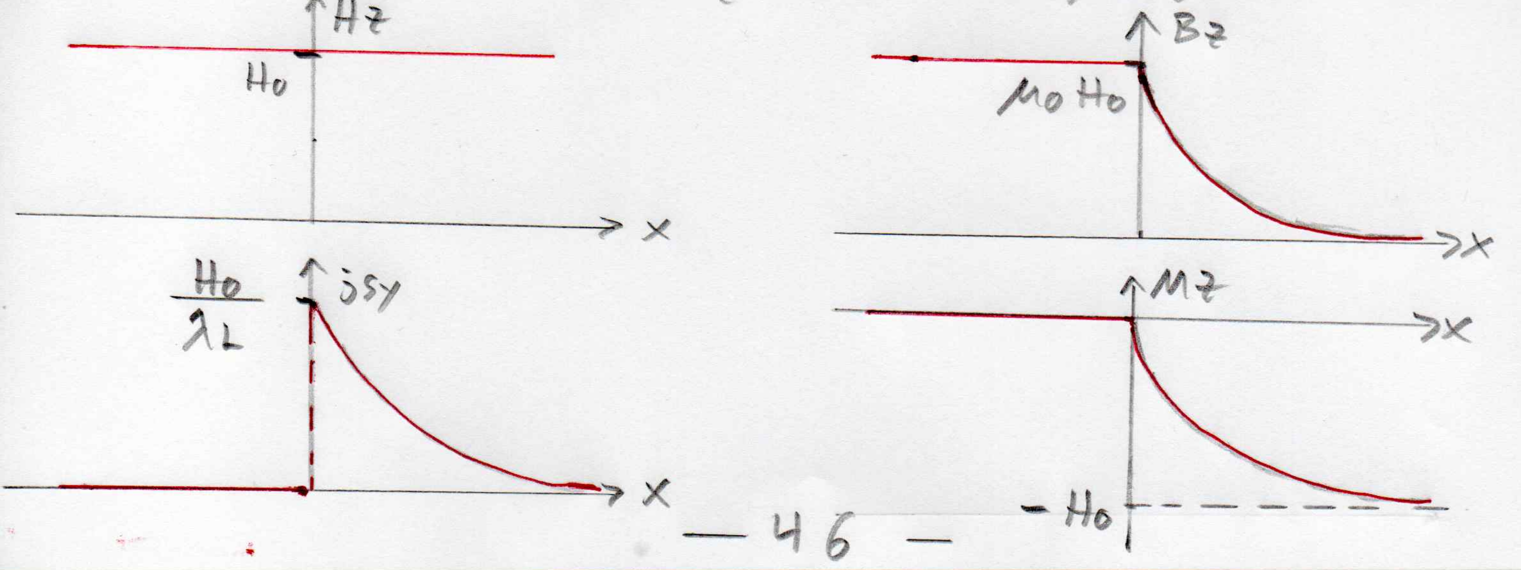
$$\vec{j}_s(x) = \frac{H_0}{\lambda_L}\, e^{-x/\lambda_L}\,\vec{e}_y, \qquad x \geq 0 \qquad (4.60)$$

And, finally, we can also determine the magnetization

$$\vec{M}(x) = \frac{\vec{B}(x)}{\mu_0} - \vec{H}(x) \qquad (4.61)$$

Inserting (4.57)–(4.49) therein we obtain

$$\vec{M}(x) = H_0\left(e^{-x/\lambda_L} - 1\right)\vec{e}_z, \quad x \geq 0 \qquad (4.62)$$



Thus, all three quantities $\vec{B}(x)$, $\vec{J}_S(x)$, and $\vec{M}(x)$ depend exponentially from the depth x in the superconductor. Their behaviour is determined by the London penetration depth λ_L.

The London theory describes self-consistently the emergence of a superconducting current at the surface of the superconductor within a skin, which is of the order of the London penetration depth λ_L. Due to this superconducting current the inside of the superconductor turns out to be an ideal diamagnet. In this way the London theory provides a quantitative description of the Meissner-Ochsenfeld effect.

4.4 Superconducting Electrons:

Let us follow for the time being the original argument of London that somehow single electrons can be superconducting. Thus, we assume

$$e_S = e \,, \quad m_S = m \tag{4.63}$$

in order to get first estimates. With this we insert (4.47) so that we can determine the density of superconducting electrons n_S from the experimentally observable London penetration depth:

$$n_S = \frac{m}{e^2 \mu_0} \cdot \frac{1}{\lambda_L^2} = 2.8 \cdot 10^{13} \frac{1}{m} \cdot \frac{1}{\lambda_L^2} \tag{4.64}$$

For low-temperature superconductors we then obtain the following values:

element	Al	Cd	In	Pb
$\lambda_L / \text{Å}$	500	1300	640	390
$n_S / 1/m^3$	$1.1 \cdot 10^{28}$	$1.7 \cdot 10^{27}$	$6.8 \cdot 10^{27}$	$1.8 \cdot 10^{28}$

Now we compare this estimate of the superconducting electron density with the corresponding density of normal conducting electrons. To this end we assume as a good approximation that each metallic atom just provides one electron for the conduction band. With this we get

$$n_n = \frac{N}{V} = \frac{M/V}{M/N} = \frac{\rho_m}{m_{at}} \tag{4.65}$$

where $\rho_m = M/V$ denotes the mass density of the metal and $\rho_{at} = M/N$ the atomic mass. For the above low-temperature superconductors we then obtain the following value for the densities of normal conducting electrons:

element	Al	Cd	In	Pb
$\varrho_m / kg/m^3$	2702	8650	7362	11340
m_{at}/u	27	112	115	207
n_n / m^3	$6 \cdot 10^{28}$	$4.6 \cdot 10^{28}$	$3.8 \cdot 10^{28}$	$3.3 \cdot 10^{28}$

By comparing both tables we can conclude that there are always more normal than superconducting electrons:

$$n_n > n_s \tag{4.66}$$

4.5 Remarks:

Let us summarise the so far obtained physical conclusions of the London theory in form of some remarks:

1) In contrast to the historic assumption of London, superconducting electrons consist of Cooper pairs. Thus, we have instead of (4.63) to deal with (4.9). Furthermore, the density of Cooper pairs amounts to one half of the density of superconducting electrons:

$$n_s = \frac{n_s'}{2} \tag{4.67}$$

with this we conclude

$$\lambda_L \overset{(4.47)}{=} \sqrt{\frac{m_s}{e_s^2 n_s \mu_0}} = \sqrt{\frac{2m}{(2e)^2 (n_s'/2)\mu_0}} = \sqrt{\frac{m}{e^2 n_s' \mu_0}} \overset{(4.47)}{=} \lambda_L' \tag{4.68}$$

We read off that the value of the London penetration depth is not affected by the Cooper pairing mechanism.

2) A normal and a superconductor differ as follows. Whereas a normal conductor is characterised by $n_s \to 0$ and, thus, $\lambda_L \to \infty$ due to (4.47), a superconductor has both a finite n_s and a finite λ_L.

3) If one apply an alternating magnetic field to a superconductor, then the skin effect of the telegraph equation and the London penetration mechanism of the extended telegraph equation (4.52) would be superimposed.

4) We remark already here that the London penetration depth λ_L, which has only a small temperature dependence, is not the only length scale characterising superconductivity. In the BCS theory we will also encounter the Cooper pair length ξ, which strongly depends on the temperature.

5) The Ginzburg-Landau theory discriminates between superconductors of type I and type II by considering the ratio of both length scales

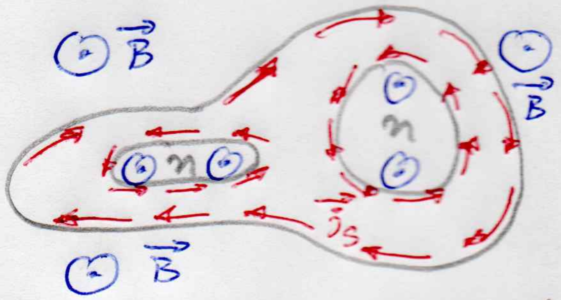
$$\varkappa = \frac{\lambda_L}{\xi} \tag{14.69}$$

Namely it turns out that for $\varkappa < 1/\sqrt{2} \approx 0.7$ and $\varkappa > 1/\sqrt{2} \approx 0.7$ type I and type II superconductors exist, respectively.

4.6 Conservation of Fluxoid and its Quantization:

So far we have considered the London theory for a simply connected superconducting region. Now we extend the discussion to a not simply connected regions of a superconductor.

4.6.1 Conservation of Flux:



According to Section 2.3 a type II superconductor has for $B_{c1}(T) < B < B_{c2}(T)$ and $T < T_c$ also normal conducting regions.

Outside of the superconductor and inside the normal conducting regions a magnetic field $\vec{B}$ is given from outside. At the surface of the superconductor and at the surface of the normal conducting regions currents of superconducting electrons flow. They cause an exponential decay of the magnetic field inside the superconducting region. Note that, due to the considered geometry, the currents at the surface of the superconductor and at the surface of the normal conducting regions flow in opposite directions.

We now revisit the induction law in its differential form provided by the Maxwell equation (M3). Applying the Stokes theorem we convert it into its corresponding integral form:

$$-\int_F \frac{\partial \vec{B}}{\partial t} \cdot d\vec{F} \overset{(M3)}{=} \int_F \operatorname{rot} \vec{E} \cdot d\vec{F} \overset{\text{Stokes}}{=} \oint_{\partial F} \vec{E} \cdot d\vec{r} \tag{14.70}$$

Here F denotes an area and ∂F stands for its boundary in form of a closed curve. Does the area F not change with time, we can pull the time derivative at the left-hand side of (4.70) outside of the area integral and yield

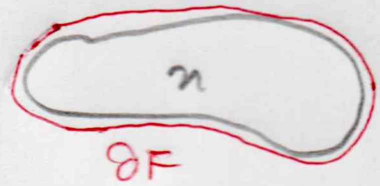
$$\frac{\partial}{\partial t}\int_F \vec{B}\cdot d\vec{F} = -\oint_{\partial F} \vec{E}\cdot d\vec{r} \qquad (4.71)$$

Thus, on the left hand side the temporal derivative of the magnetic flux through the area F appears, whereas the integral along the closed curve ∂F at the right-hand side corresponds to the induction voltage being created along ∂F. The minus sign represents the Lenz rule.

Provided that the closed curve ∂F is inside the superconductor, we can apply the London equation (L1)

$$\frac{\partial}{\partial t}\left\{\int_F \vec{B}\cdot d\vec{F} + \Lambda_s \oint_{\partial F} \vec{j}_s\cdot d\vec{r}\right\} = 0 \qquad (4.72)$$

Thus, we read off from (4.72) a conserved quantity, which is called fluxoid:



$$\frac{\partial}{\partial t}\phi' = 0 \qquad (4.73)$$

The fluxoid ϕ' has two components:

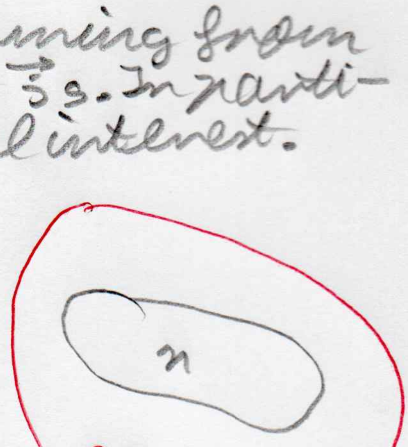
$$\phi' = \phi + \Lambda_s \oint_{\partial F} \vec{j}_s\cdot d\vec{r} \qquad (4.74)$$

Here the first component stands for the magnetic flux through the area F

$$\phi = \int_F \vec{B}\cdot d\vec{F} \qquad (4.75)$$

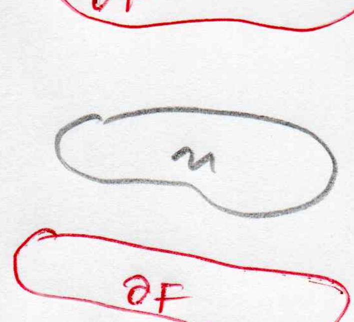
and an additional contribution stemming from the superconducting current density $\vec{j}_s$. In particular, two special cases are of physical interest.

A first special case occurs if ∂F is deep in the superconducting region. Then the superconducting current density vanishes, i.e. $\vec{j}_s = \vec{0}$, and (4.74) reduces to



$$\phi' = \phi \quad , \quad \partial F \text{ deep in superconductor} \qquad (4.76)$$

Another special case occurs if the area F is completely in the superconductor. Then the area integral in form of the magnetic flux (4.75) can be further evaluated by imposing the second London equation (L2) and the Stokes theorem:



$$\phi \overset{(4.75),(L2)}{=} -\lambda_s \int_F \mathrm{rot}\,\vec{j}_s \cdot d\vec{F} \overset{\text{Stokes}}{=} -\lambda_s \oint_{\partial F} \vec{j}_s \cdot d\vec{r} \qquad (4.77)$$

Thus, we read off from (4.74) and (4.77) that then the fluxoid vanishes:

$$\phi' = 0\,, \quad F \text{ deep in superconductor} \qquad (4.78)$$

4.6.2 Reformulation:

The definition of the fluxoid ϕ' from (4.74), (4.75) can be reformulated along the following lines. Due to the Maxwell equation (M4) and the vector field decomposition theorem of Helmholtz the magnetic induction $\vec{B}$ can be expressed by the vector potential $\vec{A}$ according to

$$\vec{B} = \mathrm{rot}\,\vec{A} \qquad (4.79)$$

Thus, taking into account (4.79) as well as (4.11) and (4.16) yields for the fluxoid (4.74), (4.75):

$$\phi' = \int_F \mathrm{rot}\,\vec{A} \cdot d\vec{F} + \frac{m_s}{e_s^2 n_s} \oint_{\partial F} e_s n_s \vec{v}_s \cdot d\vec{r} \qquad (4.80)$$

Applying the Stokes theorem to the area integral, we obtain

$$\phi' = \oint_{\partial F} \left(\vec{A} + \frac{m_s}{e_s}\vec{v}_s\right) \cdot d\vec{r} \qquad (4.81)$$

Thus, it turns out that the fluxoid can be represented by a closed curve integral, which turns out to be independent of the density of superconducting electrons n_s as a material parameter. Introducing the canonical momentum of superconducting electrons

$$\vec{P}_s = m_s \vec{v}_s + e_s \vec{A} \qquad (4.82)$$

as the sum of their kinetic momentum $m_s \vec{v}_s$ and a contribution stemming from the vector potential $\vec{A}$, we finally yield

$$\phi' = \frac{1}{e_s} \oint_{\partial F} \vec{P}_s \cdot d\vec{r} \qquad (4.83)$$

Thus, the fluxoid is determined by a closed curve integral over the canonical momentum.

4.6.3 Quantization:

Although the London theory provides a purely classical description of electrodynamic properties of a superconductor, a first step towards a quantum description is possible by invoking the semi-classical Bohr-Sommerfeld quantization. The latter was originally developed to explain the energy levels of the hydrogen atom and, thus, its spectral absorption and emission lines. It says that the phase-space volume is quantized according to

$$\oint_{\partial F} \vec{p} \cdot d\vec{r} = n h , \quad n \in \mathbb{N}_0 \tag{4.84}$$

where $\vec{p}$ denotes the canonical momentum. Applying the Bohr-Sommerfeld quantization condition (4.84) to the macroscopic fluxoid phenomenon (4.83), we get

$$\phi' = n \frac{h}{e_s} , \quad n \in \mathbb{N}_0 \tag{4.85}$$

Thus, the conserved fluxoid can only have quantized values. The smallest possible change of the fluxoid is called flux quantum

$$\phi_0 = \frac{h}{e_s} \tag{4.86}$$

According to various, seminal experiments, which we describe in the subsequent section, it turns out that this elementary flux quantum is given by

$$\phi_0 = \frac{h}{2e} \tag{4.87}$$

This implies that the superconducting electrons have the charge

$$e_s = 2e \tag{4.88}$$

in agreement with the Cooper pairs of the BCS theory. The resulting concrete value of the flux quantum reads in SI units

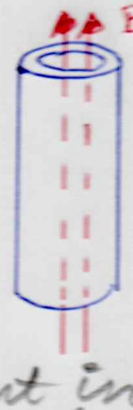
$$\phi_0 = 2.07 \cdot 10^{-15} \, T m^2 \tag{4.89}$$

Thus, we conclude that the London theory implies two microscopic quantities, namely the London penetration depth λ_L of the magnetic field, which explains the Meissner-Ochsenfeld effect, and the flux quantum ϕ_0, which provides a first substantial hint that superconductivity is a macroscopic quantum phenomenon.

4.7 Measurements of Flux Quantum:

Historically, a lattice of flux lines was predicted for a superconductor by Abrikosov in Moscow in 1957 on the basis of solving the Ginzburg-Landau theory. He received for this discovery the Nobel Prize of Physics in 2003. These lattices were visualized experimentally by Träuble and Essmann in Stuttgart in 1968, as explained in section 2.5. But already in 1961 two independent experiments showed the quantization of the fluxoid and measured the flux quantum.

4.7.1 General Idea:



Both experiments are based on the same set-up. A superconducting tube is cooled down to $T < T_c$. By switching on an external magnetic field $\vec{B}$ parallel to the pipe axis, superconducting currents are induced in the tube shell. Those currents flow forever and even prevail once the magnetic field is switched off, i.e. in the limit $\vec{B} \to \vec{0}$. According to Section 4.6 the current in the tube shell can not have any continuous value but must adjust such that the total magnetic flux turns out to be an integer of the elementary flux quantum (4.87).

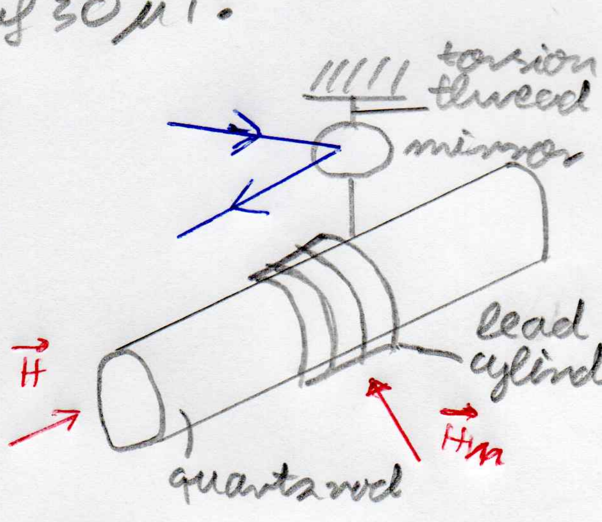
The experiments to detect flux quanta are quite challenging as the elementary flux quantum (4.89) is so small. In order to achieve a big relative change of the flux, one must try to realize states with as few flux quanta as possible. To this end it is necessary to use quite small superconducting rings in order to minimize the cross sectional area. Otherwise, the magnetic fields necessary to generate the superconducting current would also have been too small.

In 1961 the two groups of Doll and Näbauer in Munich and of Deaver and Fairbank in Stanford used quite similar experimental set ups. The tube was quite thin with a diameter of about 10 μm. The magnetic field to generate a flux quantum amounted to about

$$B = \frac{\phi_0}{\pi R^2} = \frac{2.07 \cdot 10^{-15}\,\mathrm{T\,m^2}}{\pi\,(5 \cdot 10^{-6}\,\mathrm{m})^2} = 26\,\mu\mathrm{T} \tag{4.90}$$

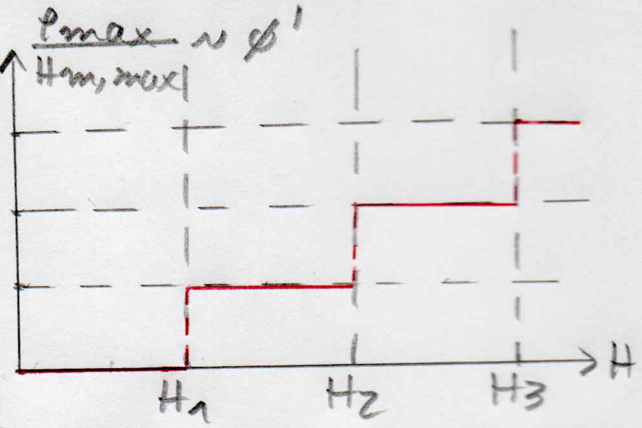
Thus, it was mandatory to shield the magnetic field of the earth, which is of the order of 30 μT.

4.7.2 Method of Doll-Näbauer:



The Munich experiment used a cylinder of lead with a width of about 1000 Å, which was evaporated on a quartz rod with a diameter of about 10 μm. Within a constant magnetic field $H < H_c$ the set up was cooled below the critical temperature of lead, i.e. $T < T_c = 7K$. As a result permanent currents were started in the cylinder of lead. After having switched off the magnetic field H, the fluxoid was frozen.

The permanent currents in the cylinder of lead represent a magnet. Due to an external periodically modulated magnetic field H_m, the particular hanging of the cylinder lead yields torsion oscillations. A resonance occurs provided that the frequency of the torsion oscillations and the frequency of the periodically modulated magnetic field H_m coincide. At resonance the amplitude of the torsion oscillations becomes that large that they can easily be measured. To this end one reflects light from a mirror which is attached at the torsion thread.



Dividing the amplitude of the torsion oscillation φ_{max} by the amplitude of the periodically modulated magnetic field $H_{m,max}$ represents the relevant experimental observable, which is proportional to the fluxoid ϕ'. It shows a staircase behaviour due to the appearance of 1, 2, 3, ... flux quanta. The respective critical magnetic fields $H_1, H_2, H_3, \ldots$, where a new flux quantum sets in, have the ratios $H_1 : H_2 : H_3 : \ldots = 1 : 2 : 3 : \ldots$ Thus, they also reveal the quantization that the magnetic flux $\mu_0 H \cdot \pi R^2$ must be an integer multiple of the flux quantum ϕ_0.

Note the experimental drawback that only about 90% of the fluxoid could be measured in this way via the magnetic flux. The remaining 10% of the fluxoid, which are realized by the superconducting current, are not observable in this experiment, see (4.74), (4.75).

4.7.3 Method of Deaver and Fairbank:

The Stanford experiment used a superconducting hollow cylinder, which is moved periodically back and forth in the longitudinal direction with a frequency of 100 Hz and an amplitude of 1 mm. At two small magnetic coils at the ends of the cylinder the resulting induction voltage was enhanced and then measured.

4.7.4 Discussion:

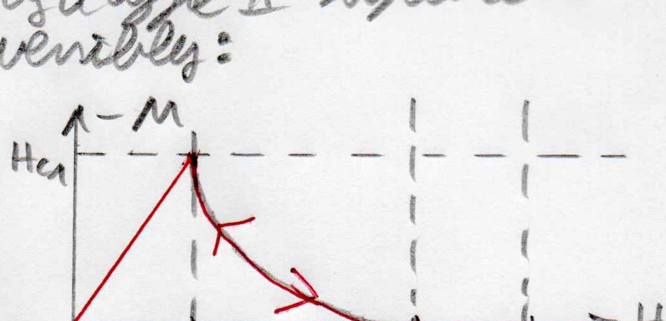
Both experiments yield the following results:

1) The magnetic flux is, apart from small corrections due to the superconducting current contribution to the fluxoid, a quantized quantity within a superconductor.
2) The measured value of the elementary flux quantum proves (4.87), i.e. the superconducting electrons must exist in pairs.
3) According to the London theory all superconducting electrons, i.e. all Cooper pairs, contribute to the nth flux quantum. Provided a transition from the nth to the n'th flux quantum occurs, all Cooper pairs together have to perform the transition.

Thus, in conclusion, the flux quantization experiments provide a strong hint that superconductivity is a macroscopic quantum phenomenon.

4.8 Structure of an Elementary Flux Quantum:

After having introduced the existence of flux quanta, we discuss now in more detail their properties. As a motivation we revisit the mixed state of type II superconductors, where flux quanta appear between the critical fields H_{c1} and H_{c2}. In case of a good single crystal the magnetization curve of a type II superconductor can be run through reversibly:

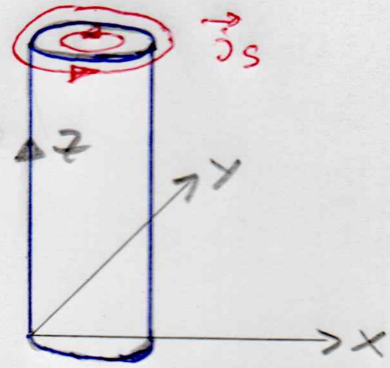


1) At H_{c1} the first flux quanta are produced. Increasing H towards H_{c2} more and more flux quanta are produced. Thus, although flux quanta turn out to repel each other, they have to move together.
2) Conversely, at H_{c2} the whole superconductor consists of flux quanta. Decreasing H towards H_{c1}, more and more flux quanta are destroyed.

Now we embark upon elucidating the structure on elementary flux quantum.

4.8.1 Inhomogeneous Helmholtz Equation:

We model an elementary flux quantum as follows:



1) The inner part represents a normal conductor in form of a delta function singularity.
2) Going outside the circular superconducting currents decrease such that in total they generate an elementary flux quantum.
3) To simplify the calculation we assume the superconducting currents to have a cylinder symmetry.

The quantization condition for an elementary flux quantum follows from (4.74), (4.75) and (4.85), (4.86):

$$\phi' = \int_F \vec{B}\cdot d\vec{F} + \Lambda_s \oint_{\partial F} \vec{j}_s \cdot d\vec{r} \stackrel{!}{=} \phi_0 \qquad (4.91)$$

Due to (4.46) we can express the abbreviation Λ_s by the London penetration length λ_L and apply the Stokes theorem to the second term. Furthermore, we introduce an area integral over a two-dimensional delta function $\delta^{(2)}(\vec{x})$ with $\vec{x} = (x,y)$:

$$\int_F \left(\vec{B} + \mu_0 \lambda_L^2 \, \mathrm{rot}\, \vec{j}_s\right)\cdot d\vec{F} \stackrel{!}{=} \phi_0 \int_F \delta^{(2)}(\vec{x})\, d\vec{F}\cdot \vec{e}_z \quad (4.92)$$

In order to obtain a single equation for the magnetic field, the superconducting current density $\vec{j}_s$ has to be eliminated. To this end we use (4.1), (4.17) and (4.44) in the stationary case:

$$\int_F \left(\vec{B} - \lambda_L^2 \Delta \vec{B}\right)\cdot d\vec{F} = \phi_0 \int_F \delta^{(2)}(\vec{x})\, d\vec{F}\cdot \vec{e}_z \qquad (4.93)$$

As this equation should be valid for any area F, we obtain a partial differential equation for the magnetic induction $\vec{B}$:

$$\Delta \vec{B} - \frac{1}{\lambda_L^2}\vec{B} = -\frac{\phi_0}{\lambda_L^2}\,\delta^{(2)}(\vec{x})\,\vec{e}_z \qquad (4.94)$$

The left-hand side corresponds to the Helmholtz equation (4.52), whereas the inhomogeneity at the right-hand side guarantees the freezing of an elementary flux quantum.

In the following we solve the inhomogeneous Helmholtz equation (4.94) step by step.

4.8.2 Simplifications:

The problem at hand can be further simplified by the following considerations. As we consider a cylinder-symmetric problem, we express the Laplace operator in (4.94) in cylinder coordinates:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{\lambda_L^2}\right)\vec{B}=-\frac{\phi_0}{\lambda_L^2}\delta^{(2)}(\vec{x})\vec{e}_z \quad (4.95)$$

Assuming that the flux quantum is perfectly cylinder-symmetric, the magnetic induction $\vec{B}$ can not depend neither on z nor on φ. This reduces (4.95) to

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{\lambda_L^2}\right)\vec{B}=-\frac{\phi_0}{\lambda_L^2}\delta^{(2)}(\vec{x})\vec{e}_z \quad (4.96)$$

As the superconducting electrons flow around a cylinder the magnetic induction $\vec{B}$ can only have a z-component:

$$\vec{B}(r)=B_z(r)\vec{e}_z \quad (4.97)$$

with the differential equation

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{\lambda_L^2}\right)B_z(r)=-\frac{\phi_0}{\lambda_L^2}\delta^{(2)}(\vec{x}) \quad (4.98)$$

Thus, it now remains to solve the differential equation (4.98). To this end we have to review some mathematical facts about cylinder functions and modified cylinder functions.

4.8.3 Cylinder Functions:

The homogeneous part of (4.98) corresponds to the Bessel differential equation

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{\lambda_L^2}-\frac{n^2}{r^2}\right)B_z(r)=0,\quad n\in\mathbb{N}_0 \quad (4.99)$$

Here the London penetration length λ_L can be eliminated by rescaling the radius and introducing the dimensionless coordinate

$$u=\frac{r}{\lambda_L} \quad (4.100)$$

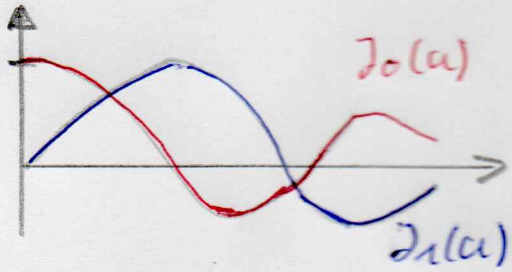
This converts (4.99) in a dimensionless Bessel differential equation

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+1-\frac{n^2}{u^2}\right)B_z(u)=0,\quad n\in\mathbb{N}_0 \quad (4.101)$$

It represents a differential equation of second order and, thus, has two fundamental solutions:

1) Bessel function (cylinder function of 1st kind) of order n: $J_n(u)$

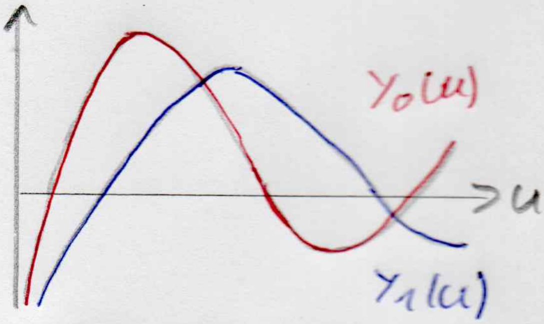
$\lim_{u \to 0} J_n(u) = \delta_{n,0}$



2) Neumann function (cylinder function of 2nd kind) of order n: $Y_n(u)$

$Y_0(u) \sim \ln u,\ u \to 0$

$Y_n(u) \sim u^{-n},\ u \to 0,\ n \in \mathbb{N}$



Note that instead of the Bessel and the Neumann functions also linear combinations of those can be used as fundamental solutions. An example is provided by the Hankel functions (cylinder functions of third kind)

$$H_n^{\pm}(u) = J_n(u) \pm i\, Y_n(u) \qquad (4.102)$$

4.8.4 Modified Cylinder Functions:

The Bessel differential equation (4.101) can be transformed such that it gets a form analogous to the original differential equation (4.98). Namely, the analytic continuation

$$u = i u' \qquad (4.103)$$

converts (4.101) into

$$\left(\frac{d^2}{du'^2} + \frac{1}{u'} \frac{d}{du'} - 1 - \frac{n^2}{u'^2} \right) B_Z(u') = 0,\ n \in \mathbb{N}_0 \qquad (4.104)$$

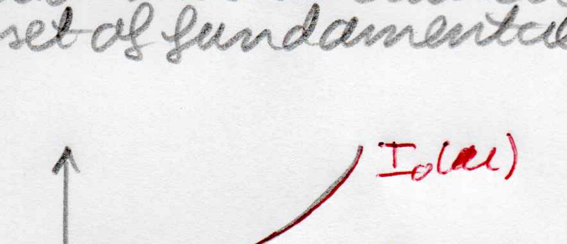
The solutions of (4.104) are obtained from an analytic continuation of the cylinder functions and are called modified cylinder functions. A set of fundamental solutions reads:

1) Modified Bessel function:

$I_n(u) = J_n(iu)\, i^{-n}$ (4.105)

$I_0(u) = \frac{1}{\sqrt{2\pi u}}\, e^u,\ u \to \infty$ (4.106)

$\lim_{u \to 0} I_n(u) = \delta_{n,0}$



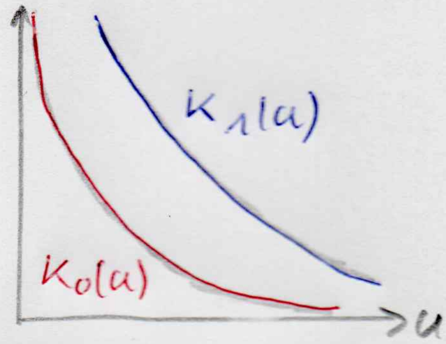
2) Modified Hankel function:

$$K_n(u) = \frac{\pi}{2} i^{n+1} H_n^{+}(iu) \quad (4.107)$$

$$K_n(u) = \sqrt{\frac{\pi}{2u}}\, e^{-u}, \quad u \to \infty$$

$$K_0(u) = -I_0(u) \ln u/2, \quad u \to 0$$

$$K_1(u) = \frac{1}{u}, \quad u \to 0$$



4.8.5 Solution of Homogeneous Helmholtz Equation:

Equipped with this mathematical knowledge we return to the inhomogeneous Helmholtz equation (4.98). In case of $r > 0$ it reduces to the homogeneous differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{\lambda_L^2} \right) B_z(r) = 0, \quad r > 0 \quad (4.108)$$

A comparison with (4.104) shows that it represents a modified Bessel differential equation of order $n = 0$. Its solution is given by a linear combination of the two fundamental solutions $I_0(r/\lambda_L)$ and $K_0(r/\lambda_L)$:

$$B_z(r) = b_1 I_0\left(\frac{r}{\lambda_L}\right) + b_2 K_0\left(\frac{r}{\lambda_L}\right) \quad (4.109)$$

However, in addition, we have to take into account the boundary condition that the magnetic field $B_z(r)$ must not diverge in the limit $r \to \infty$. Due to (4.106), the contribution of the modified Bessel function I_0 has to be excluded and (4.109) reduces to

$$B_z(r) = b\, K_0\left(\frac{r}{\lambda_L}\right) \quad (4.110)$$

Thus, we conclude that the solution of (4.108) is determined up to a proportionality constant b.

4.8.6 Proportionality Constant:

The remaining proportionality constant is now fixed by the condition that the flux is given by the elementary flux quantum ϕ_0. Due to (4.46) and (4.91) this amounts to

$$\int_F \vec{B} \cdot d\vec{F} + \mu_0 \lambda_L^2 \oint_{\partial F} \vec{j}_s \cdot d\vec{r} \stackrel{!}{=} \phi_0 \quad (4.111)$$

Here we take into account the cylinder symmetry of the problem by choosing F as a circle with radius R

and ∂F as its circumference. In order to fulfill (4.111) we proceed in the following three steps:

1) Due to (4.1), (4.17), and (M2) the superconducting current density follows from Oersted's law:

$$\vec{j}_S = \frac{1}{\mu_0} \operatorname{rot} \vec{B} \qquad (4.112)$$

Expressing the rotation in cylinder coordinates

$$\vec{j}_S = \frac{1}{\mu_0}\left\{\left(\frac{1}{r}\frac{\partial B_z}{\partial \varphi} - \frac{\partial B_\varphi}{\partial z}\right)\vec{e}_r + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\vec{e}_\varphi + \left(\frac{1}{r}\frac{\partial (r B_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial B_r}{\partial \varphi}\right)\vec{e}_z\right\} \qquad (4.113)$$

we obtain due to (4.97)

$$\vec{j}_S(r) = j_\varphi(r)\,\vec{e}_\varphi \qquad (4.114)$$

with the azimutal component

$$j_\varphi(r) = -\frac{1}{\mu_0}\frac{\partial B_z(r)}{\partial r} \underset{=}{\overset{(4.110)}{}} -\frac{b}{\mu_0 \lambda_L} K_0'\left(\frac{r}{\lambda_L}\right) \qquad (4.115)$$

due to the property of the modified Hankel function

$$K_0'(u) = -K_1(u) \qquad (4.116)$$

this then yields

$$j_\varphi(r) = \frac{b}{\mu_0 \lambda_L} K_1\left(\frac{r}{\lambda_L}\right) \qquad (4.117)$$

2) The integral over the circumference in (4.111) can now be directly performed

$$\oint_{\partial F} \vec{j}_S \cdot d\vec{r} = \int_0^{2\pi} d\varphi\, j_\varphi(R)\,\vec{e}_\varphi \cdot R\,\vec{e}_\varphi \underset{=}{\overset{(4.117)}{}} \frac{2\pi R b}{\mu_0 \lambda_L} K_1\left(\frac{R}{\lambda_L}\right) \qquad (4.118)$$

3) Correspondingly, the area integral in (4.111) is evaluated:

$$\int_F \vec{B}\cdot d\vec{F} \underset{=}{\overset{(4.97)}{}} \int_0^{2\pi} d\varphi \int_0^R dr\, r\, B_z(r)\,\vec{e}_z \cdot \vec{e}_z \qquad (4.119)$$

$$\underset{=}{\overset{(4.110)}{}} 2\pi b \int_0^R dr\, r\, K_0\left(\frac{r}{\lambda_L}\right) \underset{=}{\overset{u = r/\lambda_L}{}} 2\pi b \lambda_L^2 \int_0^{R/\lambda_L} du\, u\, K_0(u)$$

Due to the stem function

$$\int^u du'\, u' K_0(u') = -u\, K_1(u) \qquad (4.120)$$

(4.119) reduces to

$$\int_F \vec{B}\cdot d\vec{F} = 2\pi b \lambda_L^2 \left\{1 - \frac{R}{\lambda_L} K_1\left(\frac{R}{\lambda_L}\right)\right\} \qquad (4.121)$$

where we used the property (see page 59)

$$\lim_{u\to 0} u \cdot K_1(u) = 1 \qquad (4.122)$$

Inserting (4.118) and (4.121) into (4.111) yields an equation for determining the proportionality constant b, which turns out to be independent of the radius R:

$$2\pi b\lambda_L^2\left\{1-\cancel{\frac{R}{\lambda_L}K_1\left(\frac{R}{\lambda_L}\right)}\right\}+\mu_0\lambda_L^2\cdot\cancel{\frac{2\pi R b}{\mu_0\lambda_L}K_1\left(\frac{R}{\lambda_L}\right)}=\phi_0$$

$$\Longrightarrow \quad b=\frac{\phi_0}{2\pi\lambda_L^2} \qquad (4.123)$$

4.8.9 Summary:

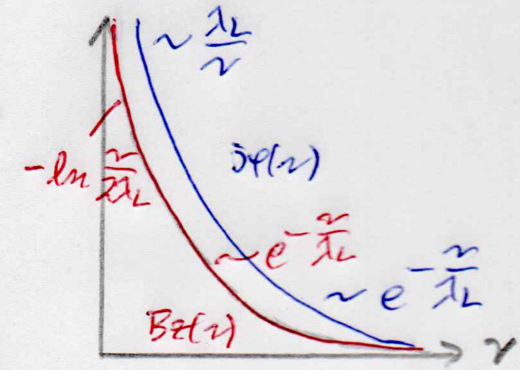
The spatial distributions of both the magnetic induction $\vec{B}$ and the superconducting current density $\vec{j}_s$ have been determined exactly without any approximation from the underlying physical assumptions. From (4.97), (4.110), and (4.123) we obtain

$$\vec{B}(r)=\frac{\phi_0}{2\pi\lambda_L^2}K_0\left(\frac{r}{\lambda_L}\right)\vec{e}_z \qquad (4.124)$$

whereas from (4.114), (4.117), and (4.123) we yield

$$\vec{j}_s(r)=\frac{\phi_0}{\mu_0 2\pi\lambda_L^3}K_1\left(\frac{r}{\lambda_L}\right)\vec{e}_\varphi \qquad (4.125)$$

With this we conclude as follows:



1) All fields vary on a length scale, which is provided by the London penetration depth λ_L.
2) The London theory yields for the magnetic field $B_z(r)$ in the limit $r\to 0$ a singularity. This represents an inconsistency as superconductivity breaks down at a finite critical field.
3) Also the superconducting current density $j_\varphi(r)$ diverges in the limit $r\to 0$.

These singularities stem from the basic assumption of the London theory that the density of superconducting electrons n_s is homogeneous through the whole superconductor. According to the BCS theory this is not the case. In particular, in the region of a flux quantum n_s turns out to be quite inhomogeneous. Such a spatial distribution of the density of superconducting electrons is taken into account within the Ginzburg-Landau theory.

4.9 Vector Potential:

In Section 4.1.6 we concluded in (4.32) that the vector potential $\vec{A}$ is proportional to the superconducting current density $\vec{j}_s$. However, this relation turns out to be valid only in a simply connected region. In a multiple connected region, as it occurs for the mixed state of a superconductor, (4.32) is no longer valid and has to be modified correspondingly.

To this end we have to go back one step and reconsider again (4.31). In fact, (4.31) determines the relation between the vector potential $\vec{A}$ and the superconducting current density $\vec{j}_s$ only up to the gradient of a scalar function χ. This modifies (4.31) according to

$$\vec{A} + \Lambda_s \vec{j}_s = \operatorname{grad} \chi \tag{4.126}$$

In the stationary case the continuity equation (M5) reduces due to (4.1) and (4.17) to

$$\operatorname{div} \vec{j}_s = 0 \tag{4.127}$$

Thus, taking the divergence of (4.126) yields

$$\operatorname{div} \vec{A} = \operatorname{div} \operatorname{grad} \chi = \Delta \chi \tag{4.128}$$

Note that the vector potential is not uniquely defined due to a possible gauge transformation. For instance, we can choose the Coulomb gauge

$$\operatorname{div} \vec{A} = 0 \tag{4.129}$$

so that the scalar function χ fulfills the Laplace equation

$$\Delta \chi = 0 \tag{4.130}$$

Due to the cylinder symmetry, both $\vec{A}$ and $\vec{j}_s$ point along the $\vec{e}_\varphi$-direction. Thus, from the gradient in cylinder coordinates

$$\operatorname{grad} \chi = \frac{\partial \chi}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial \chi}{\partial \varphi} \vec{e}_\varphi + \frac{\partial \chi}{\partial z} \vec{e}_z \tag{4.131}$$

and (4.126) we conclude that χ can only depend on the angle φ:

$$\chi = \chi(\varphi) \tag{4.132}$$

With (4.132) the Laplace equation (4.130) reduces to

$$\frac{\partial^2}{\partial \varphi^2} \chi(\varphi) = 0 \tag{4.133}$$

which is solved by

$$\chi(\varphi) = k\varphi + c \tag{4.134}$$

Inserting (4.134) in (4.131) then yields

$$\operatorname{grad} \chi = \frac{k}{r} \vec{e}_\varphi \tag{4.135}$$

The yet unknown proportionality constant k is determined from the quantization of the fluxoid. From (4.111) we obtain with (4.30) and the Stokes theorem

$$\oint_{\partial F} \left(\vec{A} + \mu_0 \lambda_L^2 \vec{j}_s \right) \cdot d\vec{r} \overset{!}{=} \phi_0 \qquad (4.136)$$

Inserting therein (4.126) by taking into account (4.46), we get for the circumference of a circle of radius R

$$\oint_{\partial F} \text{grad}\, \chi \cdot d\vec{r} \underset{=}{\overset{(4.135)}{=}} 2\pi \frac{k}{R} R \overset{!}{=} \phi_0 \qquad (4.137)$$

This determines the proportionality constant k, which turns out to be independent of the radius R:

$$k = \frac{\phi_0}{2\pi} \qquad (4.138)$$

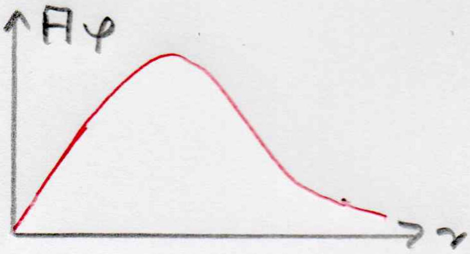
Thus, we conclude from (4.46), (4.125), (4.126), (4.135), and (4.138)

$$\vec{A} = A_\varphi \vec{e}_\varphi, \quad A_\varphi = \frac{\phi_0}{2\pi} \left\{ \frac{1}{r} - \frac{1}{\lambda_L} K_1\left(\frac{r}{\lambda_L}\right) \right\} \qquad (4.139)$$

Note that the divergence of the vector potential $\vec{A}$ in cylinder coordinates reads

$$\text{div}\, \vec{A} = \frac{1}{r} \frac{\partial (r A_r)}{\partial r} + \frac{1}{r} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} \qquad (4.140)$$

so (4.139) fulfills, indeed, the Coulomb gauge (4.129). Due to (4.122) we read off from (4.139) that the vector potential vanishes in the limit $r \to 0$ and, thus, does not reveal any singularity.



4.10 Extension of London Theory:

The London theory represents a phenomenological electrodynamic theory for superconductors. It can explain the Meissner-Ochsenfeld effect by introducing the London penetration depth λ_L. But the analysis of the elementary flux quantum ϕ_0 leads to divergencies of both the magnetic induction and the superconducting current density. These singularities make it necessary to further refine the London theory.

The basic idea in view of the Ginzburg-Landau theory is to introduce a wave function Ψ for the superconducting electrons, so that their density is given by

$$n_s = \Psi^* \Psi \qquad (14.141)$$

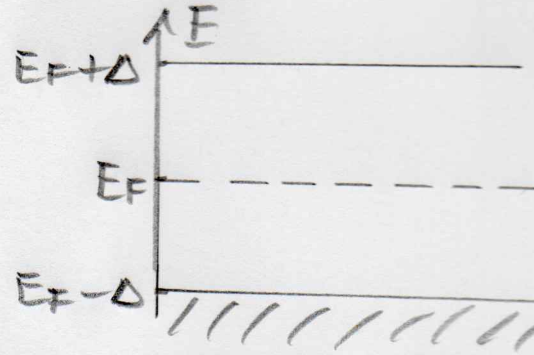
In a quantum theory one would then allow for a spatial dependence of the wave function Ψ. For a quantum theory of charged particles one obtains the current density

$$\vec{j}_s = \frac{e_s \hbar}{2 i m_s}\left(\Psi^* \vec{D}\Psi - \Psi \vec{D}\Psi^*\right) - \frac{e_s^2}{m_s}\vec{A}\,\Psi^*\Psi \qquad (14.142)$$

Thus, ignoring the spatial dependences, reduces then (14.142) together with (14.141) to

$$\vec{j}_s = -\frac{e_s^2}{m_s}\vec{A}\, n_s \qquad (14.143)$$

in agreement with the result (14.33) from the London theory. This reasoning nourishes the prospects to describe superconductors by introducing a macroscopic wave function for the superconducting electrons and by working out a corresponding quantuum theory.



From the thermodynamic measurement of the temperature dependence of the heat capacity we concluded the existence of an energy gap Δ at the Fermi edge. Such an energy gap represents an uncertainty of the energy $\Delta E = \Delta$, which is related with an uncertainty of the momentuum:

$$E = \frac{p^2}{2m} \Rightarrow \Delta = \Delta E = \frac{p}{m}\Delta p = v_F \Delta p \qquad (14.144)$$

Such a momentuum uncertainty Δp implies via the Heisenberg uncertainty relation a corresponding position uncertainty

$$\Delta p \cdot \Delta x \geq \frac{\hbar}{2} \Rightarrow \Delta x \geq \frac{\hbar}{2\Delta p} \overset{(14.144)}{=} \frac{\hbar v_F}{2\Delta} =: \xi \qquad (14.145)$$

The lower boundary of this position uncertainty is called the coherence length ξ.

Within the BCS theory it is shown that the gap energy at zero temperature $\Delta(0)$ is related to the critical temperature T_c via

$$2\Delta(0) = 3.52\, k_B T_c \qquad (14.146)$$

Thus, we conclude from (4.145) and (4.146)

$$\xi = 0.18 \frac{\hbar v_F}{k_B T_c} \qquad (4.147)$$

Depending on the Fermi velocity v_F and the critical temperature T_c of the superconductor the concrete value of the coherence length ξ varies from 5 to 10^4 Å. For instance, for the element aluminium we have the typical values

element	v_F	ξ	λ_L
Al	10^8 m/s	10^4 Å	300 Å

Thus, we obtain from (4.69) that the Ginzburg-Landau parameter amounts to $\varkappa = 0.03 < 0.71$, so that aluminium is a type I superconductor in agreement with the table in Section 2.2.

Note that, in particular, both the London penetration length λ_L and the coherence length ξ turn out to have a different dependence on the Cooper pair density n_s. For the London penetration length λ_L we read off from (4.47)

$$\lambda_L \sim \frac{1}{\sqrt{n_s}} \qquad (4.148)$$

In contrast to that we get for the coherence length ξ from (4.147)

$$\xi \sim v_F \sim n_s^{1/3} \qquad (4.149)$$

where we have neglected an additional n_s-dependence of the energy gap Δ.

Due to these different dependences on n_s it is possible that the Ginzburg-Landau parameter (4.69) obtains values, which can be both smaller and larger than the critical value $\varkappa_c = 1/\sqrt{2} \approx 0.7$, which defines the border between type I and type II superconductors.