5 Ginzburg-Landau Equations

The London theory was extended by Ginzburg and Landau in 1950 by allowing for a spatial variation of the superconducting electron density. However, both charge e_s and mass m_s of the superconducting electrons are still considered as unexplained model parameters. This problem was only solved by Gorkov in 1959, who managed to derive the phenomenological Ginzburg-Landau theory microscopically from the BCS theory and found the identification $e_s = 2e$, $m_s = 2m$ due to the formation of Cooper pairs. Furthermore, Abrikosov contributed by predicting on the basis of the Ginzburg-Landau theory a lattice of flux lines for type II superconductors. Therefore, one also speaks from the Ginzburg-Landau-Abrikosov-Gorkov (GLAG) theory. Both Ginzburg and Abrikosov received the Nobel Prize of physics in 2003, whereas Landau was already a Nobel Prize recipient in 1962.

5.1 Motivation:

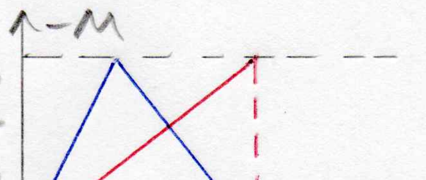
Why was it necessary to extend the London theory to the Ginzburg-Landau theory? We start with collecting arguments, which motivate this procedure.

5.1.1 Flux Quantum:

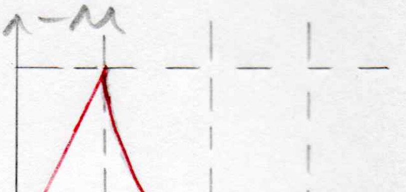
As already explained in Chapter 4, the London theory yields at the center of a flux quantum divergencies for both the magnetic induction and the superconducting current density. Thus, a more refined theory is needed, which predicts better results for the spatial profiles of both $\vec{B}(\vec{r})$ and $\vec{j}_s(\vec{r})$ at the flux quantum in a superconductor.

5.1.2 Magnetization Curves:

The London theory fails to predict the critical fields of type I and type II superconductors. However, their knowledge is quite decisive for many applications. Additionally,



type I



type II

the precise shape of the magnetization curve of the mixed state of type II superconductors is not described within the London theory.

5.1.3 Flux Penetration:

The penetration of flux quanta in a superconductor is different for type I and type II:

type I | type II

Millions of flux quanta penetrate macroscopically the superconductor and yield a meandering structure.

single flux quanta penetrate separately the superconductor, typically forming a hexagonal flux lattice.

This difference between type I and type II superconductors needs a theoretical explanation.

5.1.4 Surface Energy:

There are two different terms contributing to the surface energy of a superconductor:

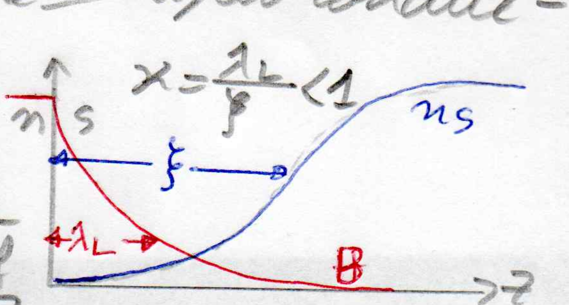
1) Within a depth of the length scale ξ Cooper pairs have to be broken. This leads to a loss of energy per area, which is of the order
$$\gamma_1 = \left(\frac{E}{F}\right)_1 = \xi \frac{B_c^2}{2\mu_0} \qquad (5.1)$$

2) The magnetic field can penetrate the superconductor up to a depth, whose order is determined by the London penetration length λ_L. According to the discussion of the thermodynamic properties of superconductors in Chapter 3 at (3.25) this yields an energy gain per area, which is of the order
$$\gamma_2 = \left(\frac{E}{F}\right)_2 = -\lambda_L \frac{B_c^2}{2\mu_0} \qquad (5.2)$$

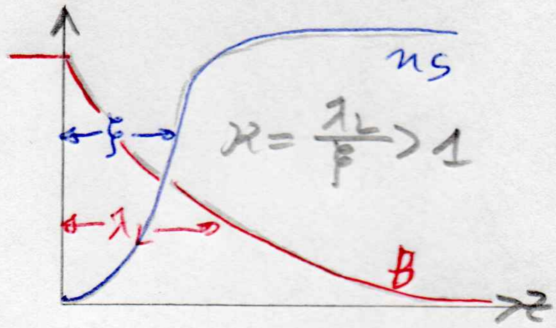
Combining both contributions yields the surface energy
$$\gamma = \gamma_1 + \gamma_2 \overset{(5.1),(5.2)}{=} (\xi - \lambda_L)\frac{B_c^2}{2\mu_0} \begin{cases} >0 \; ; \; \xi > \lambda_L \\ <0 \; ; \; \xi < \lambda_L \end{cases} \qquad (5.3)$$

This has the physical consequence to explain a major difference between type I and type II superconductors:

1) Type I superconductor ($\gamma > 0$):
The generation of s n-boundaries costs energy. Therefore, a homogeneous stable state is preferred. Only small numbers of sn-boundaries are formed.



2) Type II superconductors ($\gamma < 0$): The generation of SN-boundaries releases energy. Therefore, an inhomogeneous state is preferred. As many SN-boundaries as possible are generated.



5.1.5 Ginzburg-Landau-Parameter:

The Ginzburg-Landau theory contains a single material parameter in form of the Ginzburg parameter

$$\varkappa = \frac{\lambda_L}{\xi} \tag{5.4}$$

A more refined derivation within the Ginzburg-Landau theory reveals that the difference between type I and type II superconductors does not occur at $\varkappa_c = 1$ according to the simplified considerations in subsection 5.1.4. Instead the boundary occurs at

$$\varkappa_c = \frac{1}{\sqrt{2}} \approx 0.7 \tag{5.5}$$

with $\varkappa < \varkappa_c$ ($\varkappa > \varkappa_c$) corresponding to type I (type II) superconductors. Exemplary values of the Ginzburg-Landau parameter for different materials are

material	Al	Nb_3Sn	high T_c
$\varkappa$	0.03	30 - 100	several hundred
	type I	type II	

5.2 Homogeneous Superconductor:

At first, we work the Landau theory for a homogeneous superconductor. It is valid in the absence of an external magnetic field and describes phenomenologically the second-order phase transition between a normal conductor and a superconductor. Note that this Landau theory is also applicable to other types of second-order phase transitions. The reason for this general applicability of the Landau theory is the universality of second-order phase transitions. This means that second-order phase transitions with the same dimensionality and the same number of order parameters behave in the vicinity of the critical point in the same way, i.e. they reveal the same critical exponents for the same observables.

5.2.1 Postulates:

In the following we list the postulates of the Landau theory for homogeneous superconductors:

1) The complex function $\Psi = \Psi(T)$ of the temperature T represents an order parameter within the general context of phase transitions. The value of Ψ then indicates whether a material is superconducting or in the normal conducting phase:

$$\begin{aligned} &\text{superconducting phase:} && \Psi(T<T_c) \neq 0 \\ &\text{normal conducting phase:} && \Psi(T \geq T_c) = 0 \end{aligned} \qquad (5.6)$$

2) The physical interpretation of the order parameter Ψ is provided by the fact that its absolute square represents the density of superconducting electrons

$$n_s(T) = \Psi^*(T)\,\Psi(T) \qquad (5.7)$$

3) As we can neglect changes of the volume V for a superconductor, the dependence of the free enthalpy G_s from the volume V can be ignored. Furthermore, we ignore in this section the magnetic field. Therefore, the free enthalpy only depends on the order parameter Ψ and the temperature T:

$$G_s = G_s(\Psi, T) \qquad (5.8)$$

4) In the vicinity of the phase transition at $T = T_c$ the order parameter Ψ is small due to (5.6). Therefore, the free enthalpy G_s can be expanded in powers of this small order parameter Ψ. But, as the free enthalpy G_s is real, it can only be Taylor expanded in powers of the density of superconducting electrons (5.7). Thus we obtain for the free enthalpy per volume $g_s = G_s/V$

$$g_s(\Psi^*, \Psi, T) = g_n(T) + \alpha(T)\,|\Psi|^2 + \frac{1}{2}\beta(T)\,|\Psi|^4 + \dots \qquad (5.9)$$

5) For the temperature dependence of the Landau coefficients $\alpha(T), \beta(T)$ in (5.9) we assume

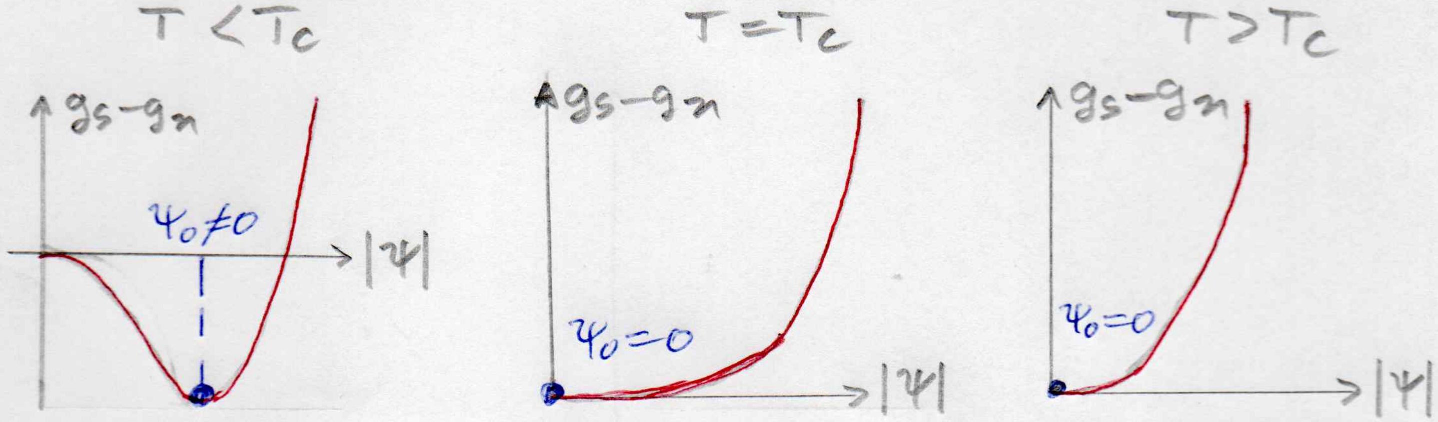
$$\alpha(T) = (T - T_c)\,\alpha_c', \qquad \alpha_c' = \frac{d\alpha(T)}{dT}\bigg|_{T=T_c} > 0 \qquad (5.10)$$

$$\beta(T) = \beta > 0 \quad \text{for all } T \qquad (5.11)$$

These postulates guarantee the occurrence of a second-order phase transition as we show in the following.

5.2.2 Qualitative Discussion:

We already obtain a qualitative understanding of the second-order phase transition by depicting the free enthalpy density difference $g_s - g_n$ as a function of the order parameter Ψ for different signs of the Landau parameter α.



A minimum $\Psi_0 \neq 0$ exists. Thus, an ordered state has emerged, which leads to an energy density gain.

Due to a critical slowing down the minimum $\Psi_0 = 0$ is only reached after a longer time.

The energetic minimum occurs at $\Psi_0 = 0$. Thus, no ordered state has emerged.

Let us investigate this qualitative finding now more quantitative.

5.2.3 Equilibrium States:

An equilibrium state has the property that it extremizes the free enthalpy density with respect to a variation of the order parameter:

$$\frac{\partial g_s}{\partial \Psi^*} = 0 \qquad (5.12)$$

Inserting (5.09) in (5.12) yields

$$\Psi\left[\alpha(T) + \beta(T)|\Psi|^2 + \ldots\right] = 0 \qquad (5.13)$$

Thus, we obtain two kinds of equilibrium states as solutions from (5.13):

1) $\Psi_0 = 0$ for all T $\qquad (5.14)$

2) $|\Psi_0| = \sqrt{-\frac{\alpha(T)}{\beta}}$ for $T < T_c$ $\qquad (5.15)$

5.2.4 Stability of Equilibrium States:

Inserting the equilibrium states (5.14), (5.15) into the free enthalpy density (5.9) yields

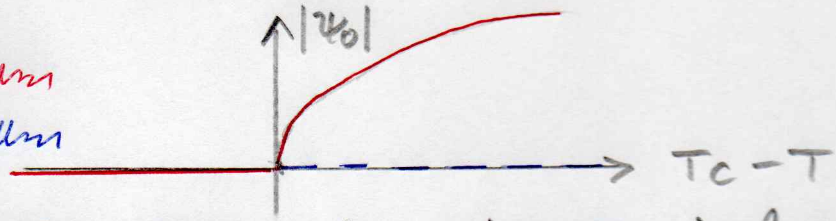
1) $g_s(\Psi_0 = 0) = g_n$ for all T $\qquad (5.16)$

2) $g_s(\Psi_0 \neq 0) = g_n - \frac{\alpha(T)^2}{2\beta}$ for all $T < T_c$ $\qquad (5.17)$

Always that equilibrium state is realised, which corresponds to a smaller enthalpy density. Thus, we conclude from (5.16), (5.17) by taking into account (5.10), (5.11):

— stable equilibrium

— unstable equilibrium



The phase transition occurring at $T = T_c$ is characterised by

1) Symmetry breaking:
Although the free enthalpy density (5.9) possesses the U(1)-symmetry

$$\psi, \psi^* \longrightarrow \psi e^{i\varphi}, \psi^* e^{-i\varphi} \tag{5.18}$$

at $T < T_c$ an equilibrium state $\psi_0 = |\psi_0| e^{i\varphi_0}$ is realised, which no longer has this U(1)-symmetry.

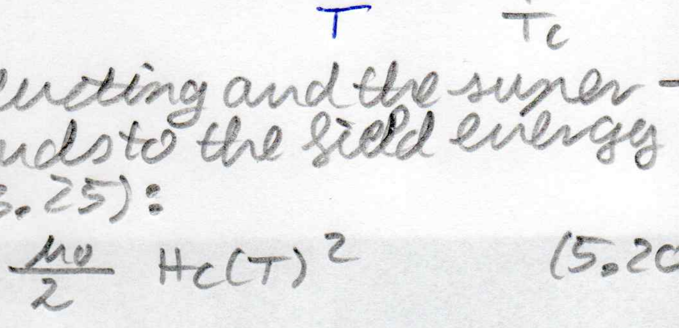
2) Critical slowing down:
For $T \to T_c$ the potential landscape $g_s(|\psi|)$ becomes flatter and flatter. This corresponds to stronger fluctuations of the order parameter around $\psi = 0$.

5.2.5 Condensation Energy Density:

From (5.17) we read off that the superconducting state has a smaller free enthalpy density than the normal conducting state. This difference in the respective free enthalpies per volume is called condensation energy density:

$$\Delta g = g_n - g_s \overset{(5.17)}{=} \frac{\alpha(T)^2}{2\beta} \tag{5.19}$$

On the other hand, within the discussion of the thermodynamic properties of a superconductor in Chapter 3, we showed that this condensation energy density in form of the difference of the free enthalpy densities of the normal conducting and the superconducting phase corresponds to the field energy per volume according to (3.25):



$$\Delta g = g_n - g_s = \frac{\mu_0}{2} H_c(T)^2 \tag{5.20}$$

Equating both expressions (5.19) and (5.20) allows to determine the temperature dependent critical magnetic field $H_c(T)$:

$$\frac{\alpha(T)^2}{2\beta} \overset{(5.10)}{=\!=} \frac{(T-T_c)^2 \alpha_c'^2}{2\beta} \overset{!}{=} \frac{\mu_0}{2} H_c^2(T)$$

$$\Rightarrow \quad H_c(T) = \frac{\alpha_c'}{\sqrt{\beta\mu_0}}(T_c - T), \quad T \leq T_c \qquad (5.21)$$

A direct comparison with the Kok approximation (1.3) shows that (5.21) is already quite reasonable:

$$H_c(T) = H_c(0)\left(1 - \frac{T^2}{T_c^2}\right) = H_c(0)\frac{(T_c-T)(T_c+T)}{T_c^2} \overset{T \approx T_c}{\approx} \frac{2H_c(0)(T_c - T)}{T_c^2} \qquad (5.22)$$

Thus, the material parameters of the Kok approximation (1.3), i.e. $H_c(0)$ and T_c, correspond to the material parameters T_c, α_c' and β of the Landau ansatz (5.9) – (5.11) for the free enthalpy density.

5.2.6 Remarks:

1) The Ginzburg – Landau theory is not microscopic but phenomenological. This is due to the material parameters $\alpha(T)$, β in (5.9).

2) In equilibrium we read off for the superconducting electron density

$$n_s \overset{(5.7),\,(5.15)}{=\!=\!=} -\frac{\alpha(T)}{\beta} \overset{(5.10)}{=\!=} \frac{\alpha_c'}{\beta}(T_c - T) \qquad (5.23)$$

According to the London theory the superconducting electron density n_s is related to the London penetration depth λ_L via (4.47), yielding

$$\lambda_L \overset{(4.47)}{\sim} \frac{1}{\sqrt{n_s}} \overset{(5.23)}{\sim} \frac{1}{\sqrt{T_c - T}} \qquad (5.24)$$

Thus, approaching the critical point, i.e. $T \uparrow T_c$, the London penetration depth λ_L diverges, i.e. $\lambda_L \to \infty$. The power-law behaviour

$$\lambda_L \sim \frac{1}{(T_c - T)^\nu} \qquad (5.25)$$

corresponds, according to (5.24) to the mean-field exponent $\nu_{MF} = 0.5$.

3) However, experimentally, one measures the expo-

nent $\nu = 0.63$. The deviation between ν and ν_{MF} occurs as the mean-field theory of Landau neglects any fluctuations of the order parameter Ψ.

5.3 Inhomogeneous superconductor:

The Landau theory of the previous section neglected all magnetic quantities like $\vec{B}$, $\vec{H}$, $\vec{j}_s$ in the free enthalpy. Thus, in particular, the phenomenon of flux quantization could not be described. Furthermore, to be more realistic, all physical quantities should acquire a spatial dependence. Only then it becomes possible to deal with inhomogeneities as, for instance, the occurrence of a flux quantum. Both extensions are provided by the Ginzburg-Landau theory for inhomogeneous superconductors.

5.3.1 Magnetic Energy:

The magnetic induction $\vec{B} = \mu_0 (\vec{H} + \vec{M})$ contains both the contribution of the external magnetic field $\vec{H}$ and the magnetisation $\vec{M}$ induced by the superconductor. In order to consider only the magnetic energy, which is stored in the superconductor, we consider the magnetic energy density

$$g_1 = \frac{1}{2\mu_0} (\vec{B} - \mu_0 \vec{H})^2 \qquad (5.26)$$

This corresponds to the energy density of the superconducting electron currents as they finally build up the magnetisation.

5.3.2 Kinetic Energy:

In classical mechanics the kinetic energy per particle is given by

$$E_{cl} = \frac{1}{2} m_s \vec{v}_s^{\,2} \qquad (5.27)$$

Here the momentum $m_s \vec{v}_s$ of the superconducting electron is related to the canonical momentum $\vec{P}_s$ according to (4.82), involving the vector potential $\vec{A}$. Thus, inserting (4.82) in (5.27) yields

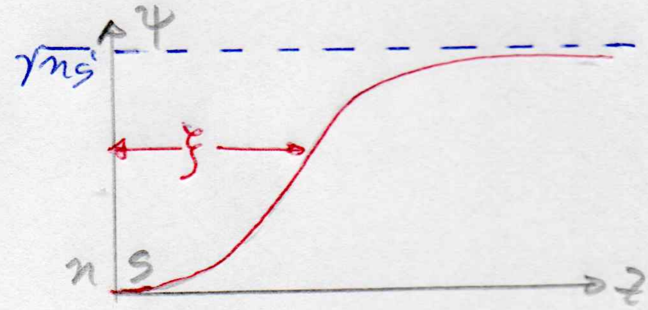
$$E_{cl} = \frac{1}{2m_s} (\vec{P}_s - e_s \vec{A})^2 \qquad (5.28)$$

In formal analogy to quantum mechanics (5.28) corresponds to the free enthalpy density

$$g_s = \frac{1}{2m_s} \left| \left(\hat{\vec{P}}_s - e_s \vec{A} \right) \Psi \right|^2 \qquad (5.29)$$

where the canonical momentum operator $\hat{\vec{P}}_s$ is identified with

$$\hat{\vec{P}}_s = \frac{\hbar}{i} \vec{\nabla} \qquad (5.30)$$



The expression (5.29), (5.30) for the kinetic energy density guarantees that a spatial variation of the order parameter, i.e. $\vec{\nabla}\Psi \neq \vec{0}$, cost energy. This property is decisive at the surface of a superconductor as there the order parameter varies from the vanishing value of the normal conductor to the bulk value $\sqrt{n_s}$ on a length scale, which is of the order of the coherence length ξ.

5.3.3 Investigation of Kinetic Energy:

Inserting (5.30) in (5.29) yields the expression

$$g_2 = \frac{1}{2m_s} \left(i\hbar \vec{\nabla} \Psi^* - e_s \vec{A} \Psi^* \right) \left(-i\hbar \vec{\nabla} \Psi - e_s \vec{A} \Psi \right) \qquad (5.31)$$

Multiplying both brackets we obtain

$$g_2 = \frac{\hbar^2}{2m_s} \vec{\nabla}\Psi^* \cdot \vec{\nabla}\Psi - \frac{\hbar e_s}{2i m_s} \left(\Psi^* \vec{\nabla} \Psi - \Psi \vec{\nabla} \Psi^* \right) + \frac{e_s^2 \vec{A}^2}{2m_s} \Psi^* \Psi \qquad (5.32)$$

Here we recognize another formal analogy to quantum mechanics, where the probability current density for a charged particle reads

$$\vec{j} = \frac{\hbar}{2im} \left(\Psi^* \vec{\nabla} \Psi - \Psi \vec{\nabla} \Psi^* \right) - \frac{e}{m} \vec{A} \Psi^* \Psi \qquad (5.33)$$

Namely, in the Ginzburg-Landau theory, the superconducting current density (4.142) appears. Taking into account (4.142) we obtain that (5.32) reduces to three terms:

$$g_2 = \underbrace{\frac{\hbar^2}{2m_s} \left| \vec{\nabla} \Psi \right|^2}_{=1} - \underbrace{\vec{j}_s \cdot \vec{A}}_{=2} - \underbrace{\frac{e_s^2}{2m_s} \vec{A}^2 |\Psi|^2}_{=3} \qquad (5.34)$$

Each of the three terms allows for a physical interpretation:

Term 1 describes spatial inhomogeneities of the order parameter Ψ. The larger these inhomogeneities, the larger the corresponding energy costs become due to 1.

Term ② represents the energy of a superconducting current density in a vector potential as is further worked out, for instance, in Landau-Lifshitz, Volume II. Due to the minus sign it is even energetically preferable to generate superconducting currents in a vector potential. As a concrete example we have analyzed in chapter 4 the properties of a flux quantum and found in (4.125) and (4.139) the corresponding profiles of the superconducting current density $\vec{j}_s$ and the vector potential $\vec{A}$, respectively. In that case we have $\vec{j}_s \cdot \vec{A} > 0$, so ② describes an energy gain.

According to the second London equation (4.32) we could identify the velocity of the superconducting electrons via (4.34). With this term ③ is recognized to be

$$\frac{1}{2} m_s \vec{v}_s^{\,2} n_s \overset{(4.34)}{=\!=} \frac{m_s}{2} \frac{e_s^2}{m_s^2} \vec{A}^2 n_s \overset{(5.7)}{=\!=} \frac{e_s^2}{2m_s} \vec{A}^2 |\psi|^2 \quad (5.35)$$

and corresponds to the kinetic energy of the superconducting electrons. Due to the minus sign in (5.34) it is even preferable to have $n_s \neq 0$ within a vector potential.

Furthermore, we note that the kinetic energy (5.32) reduces in the special case of a spatially constant order parameter, i.e. $\psi = \text{const.}$, to the corresponding one of the London theory

$$g_2(\psi = \text{const.}) = \frac{e_s^2}{2m_s} \vec{A}^2 |\psi|^2 \;\hat{=}\; (5.35) \qquad (5.36)$$

5.3.4 Free Enthalpy:

Now we are in the position to write down the total free enthalpy of an inhomogeneous superconductor. To this end, we add the contributions of the Landau theory (5.9), the magnetic energy (5.26), and the kinetic energy (5.34) to obtain the free enthalpy density

$$g_s = g_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{2m_s}|\vec{\nabla}\psi|^2 - \vec{j}_s \cdot \vec{A} - \frac{e_s^2}{2m_s}\vec{A}^2|\psi|^2$$
$$+ \frac{1}{2\mu_0}\left(\vec{B} - \mu_0 \vec{H}\right)^2 \qquad (5.37)$$

An integration over the whole volume V of the superconductor yields then the corresponding free enthalpy

$$G_s = G_n + \int_V dV \left\{ \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{2m_s}|\vec{\nabla}\psi|^2 - \vec{j}_s \cdot \vec{A} \right.$$
$$\left. - \frac{e_s^2}{2m_s}\vec{A}^2|\psi|^2 + \frac{1}{2\mu_0}\left(\vec{B} - \mu_0\vec{H}\right)^2 \right\} \qquad (5.38)$$

But one could also rewrite the kinetic energy part in (5.38) more compactly by using (5.31) instead of (5.34):

$$G_s = G_n + \int_V dV \left\{ \alpha |\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{2m_s}\left|\left(-i\hbar\vec{\nabla} - e_s\vec{A}\right)\Psi\right|^2 + \frac{1}{2\mu_0}\left(\vec{B} - \mu_0\vec{H}\right)^2 \quad (5.39)$$

This formulation clarifies that the free enthalpy of an inhomogeneous superconductor G_s represents a functional of $\Psi^*, \Psi, \vec{A}$:

$$G_s = G_s\left[\Psi^*, \Psi, \vec{A}\right] \qquad (5.40)$$

Thus, in order to obtain the corresponding Ginzburg-Landau equations, we have to perform the respective functional derivatives of G_s with respect to $\Psi^*, \Psi, \vec{A}$ in view of an extremization.

5.3.5 Variation With Respect to Order Parameter:

Concerning the variation of (5.39) with respect to Ψ^* we can immediately drop the magnetic field energy as it does not depend on the order parameter. Furthermore, we perform now quite carefully the variation with respect to Ψ^* in order to discriminate between volume and surface contributions. Namely, it turns out that they lead to volume and surface equilibrium conditions, which both have to be taken into account for a superconductor.

At first, a variation of (5.39) with respect to Ψ^* yields

$$\delta G_s = \int_V dV \left\{ \left(\alpha + \beta|\Psi|^2\right)\Psi\,\delta\Psi^* + \frac{1}{2m_s}\left(-i\hbar\vec{\nabla} - e_s\vec{A}\right)\Psi \cdot \delta\left(i\hbar\vec{\nabla} - e_s\vec{A}\right)\Psi^* \right\} \quad (5.41)$$

In order to simplify the calculation, we introduce the momentum of the superconducting electrons

$$\vec{P}_s = -i\hbar\vec{\nabla}\Psi - e_s\vec{A}\Psi \qquad (5.42)$$

For the last term in (5.41) we thus obtain

$$\int_V dV \frac{\vec{P}_s}{2m_s}\cdot\delta\left(i\hbar\vec{\nabla}\Psi^* - e_s\vec{A}\Psi^*\right) = \int_V dV \frac{\vec{P}_s}{2m_s}\cdot\left(i\hbar\vec{\nabla}\delta\Psi^* - e_s\vec{A}\,\delta\Psi^*\right) \quad (5.43)$$

A partial integration of the first term yields together with the Gauß theorem

$$\oint_{\partial V} \frac{i\hbar}{2m_s}\vec{P}_s\,\delta\Psi^* \cdot d\vec{F} - \int_V dV \underbrace{\left(\frac{i\hbar}{2m_s}\operatorname{div}\vec{P}_s + \frac{e_s}{2m_s}\vec{A}\cdot\vec{P}_s\right)}_{= \frac{1}{2m_s}\left(i\hbar\vec{\nabla} + e_s\vec{A}\right)\vec{P}_s}\delta\Psi^* \quad (5.44)$$

Substituting back the momentum of the superconducting electrons (5.42), we finally get for the variation of the free enthalpy

$$\delta G_s = \int_V dV \left\{ (\alpha + \beta|\Psi|^2)\Psi\,\delta\Psi^* + \delta\Psi^* \frac{1}{2m_s}\left(-i\hbar\vec{\nabla} - e_s\vec{A}\right)^2 \Psi \right.$$
$$- \frac{i\hbar}{2m_s} \oint_{\partial V} \delta\Psi^* \left(i\hbar\vec{\nabla} + e_s\vec{A}\right)\Psi \cdot d\vec{F} \qquad (5.45)$$

This variation vanishes provided the following two conditions are fulfilled:

volume V: $\alpha\Psi + \beta|\Psi|^2\Psi + \frac{1}{2m_s}\left(-i\hbar\vec{\nabla} - e_s\vec{A}\right)^2\Psi = 0$ (5.46)

boundary ∂V: $\left(i\hbar\vec{\nabla} + e_s\vec{A}\right)\Psi \cdot \vec{n} = 0$ (5.47)

Here $\vec{n}$ denotes the normal vector of the surface ∂V. In a similar way a variation of the free enthalpy (5.39) with respect to Ψ yields the complex conjugate of the conditions (5.46), (5.47):

volume V: $\alpha\Psi^* + \beta|\Psi|^2\Psi^* + \frac{1}{2m_s}\left(-i\hbar\vec{\nabla} - e_s\vec{A}\right)^2\Psi^* = 0$ (5.48)

boundary ∂V: $\left(-i\hbar\vec{\nabla} + e_s\vec{A}\right)\Psi^* \cdot \vec{n} = 0$ (5.49)

5.3.6 Volume Condition:

On the one hand, it looks like as if the volume condition (5.46) could be interpreted as an eigenvalue problem of the type

$$\hat{H}\Psi = -\alpha\Psi \qquad (5.50)$$

This observation underlines the formal analogy between the Ginzburg-Landau theory and quantum mechanics.

On the other hand, apart from differences in the physical interpretation, there are two essential discrepancies between the Ginzburg-Landau theory and quantum mechanics:

1) The operator $\hat{H}$ in the Ginzburg-Landau theory (5.50) is nonlinear due to the term $\beta|\Psi|^2\Psi$ in (5.46). This non-linear term is important in order to describe the second-order phase transition as was explained in Section 5.2 in the context of the Landau theory for homogeneous superconductors. In contrast to that operators in quantum mechanics are linear.
2) In quantum mechanics one determines eigenvalues. But in the Ginzburg-Landau theory the phenomenological parameter α is given with its temperature dependence (5.10), thus here one has to determine, conversely, the order parameter Ψ, Ψ^* and the vector potential $\vec{A}$ such that they fit to (5.50).

5.3.7 Surface Conditions:

We combine now both surface conditions (5.47), (5.49) in order to physically interpret them. To this end we consider:

$$-\frac{e_s}{2m_s}\left[\Psi^*(5.47)+\Psi(5.49)\right]:\left[\frac{i\hbar e_s}{2m_s}\left(\Psi\vec{\nabla}\Psi^*-\Psi^*\vec{\nabla}\Psi\right)-\frac{e_s^2}{m_s}\vec{A}|\Psi|^2\right]\cdot\vec{n}=0 \qquad (5.51)$$

Comparing (5.51) with the superconducting current density (4.142), we obtain

$$\vec{j}_s\cdot\vec{n}=0 \qquad (5.52)$$

Thus, the surface conditions prescribe that superconducting electrons are not allowed to flow out of the superconductor.

5.3.8 Variation with Respect to Vector Potential:

Concerning the variation of (5.39) with respect to $\vec{A}$, we can immediately drop the Landau energy density as it does not depend on the vector potential. And for the magnetic energy density term we have to take into account that the magnetic induction depends via (4.30) from the vector potential.

With this a variation of (5.39) with respect to $\vec{A}$ yields at first

$$\delta G_s=\int_V dV\left\{-\frac{e_s}{2m_s}\left[\Psi\left(i\hbar\nabla\Psi^*-e_s\vec{A}\Psi^*\right)+\Psi^*\left(-i\hbar\nabla\Psi-e_s\vec{A}\Psi\right)\right]\cdot\delta\vec{A}\right.$$
$$\left.+\frac{1}{\mu_0}\left(\vec{B}-\mu_0\vec{H}\right)\cdot\mathrm{rot}\,\delta\vec{A}\right. \qquad (5.53)$$

In a similar way as we have applied the Gauß theorem in order to perform a partial integration of the "grad"-term in section 5.3.5, we have now to perform a partial integration with the "rot"-term in (5.53). To this end we note the vector analytic identity

$$\mathrm{div}(\vec{u}\times\vec{v})=\partial_i\varepsilon_{ijk}(u_jv_k)=\varepsilon_{ijk}\left[(\partial_iu_j)v_k+u_j(\partial_iv_k)\right]$$
$$=v_k\varepsilon_{kij}\partial_iu_j-u_j\varepsilon_{jik}\partial_iv_k=\vec{v}\cdot\mathrm{rot}\,\vec{u}-\vec{u}\cdot\mathrm{rot}\,\vec{v} \qquad (5.54)$$

Identifying $\vec{u}=\vec{B}-\mu_0\vec{H}$ and $\vec{v}=\delta\vec{A}$, the last term in (5.53) can be recast into

$$-\int_V \mathrm{div}\left[\frac{1}{\mu_0}\left(\vec{B}-\mu_0\vec{H}\right)\times\delta\vec{A}\right]+\int_V dV\,\delta\vec{A}\cdot\frac{1}{\mu_0}\mathrm{rot}\left(\vec{B}-\mu_0\vec{H}\right) \qquad (5.55)$$

In the first term in (5.55) we apply the Gauß theorem as already planned and in the second term we use the Oersted law

$$\mathrm{rot}\,\vec{B}=\mu_0\vec{j}_s \qquad (5.56)$$

Note that in the latter case we have taken into account the stationarity condition that the current density $\vec{j}$ stems exclusively from the superconducting electron density $\vec{j}_s$. This yields from (5.55)

$$= -\oint_{\partial V} \frac{1}{\mu_0}\left[(\vec{B}-\mu_0\vec{H})\times\delta\vec{A}\right]\cdot d\vec{F} + \int_V \delta\vec{A}\cdot\vec{j}_s \qquad (5.57)$$

Inserting (5.57) for the last term in (5.53) then gives

$$\delta G_s = \int_V dV \left\{ \frac{i\hbar e_s}{2m_s}\left(\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) + \frac{e_s^2}{m_s}\vec{A}|\psi|^2 + \vec{j}_s \right\}\delta\vec{A}$$
$$+ \oint_{\partial V} \delta\vec{A}\cdot\left[\frac{1}{\mu_0}(\vec{B}-\mu_0\vec{H})\times d\vec{F}\right] \qquad (5.58)$$

This variation vanishes provided the following two conditions are fulfilled:

volume V: $$\vec{j}_s = \frac{i\hbar e_s}{2m_s}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right) - \frac{e_s^2}{m_s}\vec{A}|\psi|^2 \qquad (5.59)$$

boundary ∂V: $$(\vec{B}-\mu_0\vec{H})\times\vec{n} = \vec{0} \qquad (5.60)$$

Thus, with the volume condition (5.59) we have recovered the definition of the superconducting current density (4.142). And the boundary condition (5.60) means that the tangential components of the magnetic induction $\vec{B}$ and the magnetic field strength $\vec{H}$ are continuous at the volume boundary ∂V:

$$\vec{B}_{tang.} = \mu_0 \vec{H}_{tang.} \qquad (5.61)$$

Note that the latter we have already anticipated and used in Section 4.3 at (4.59).

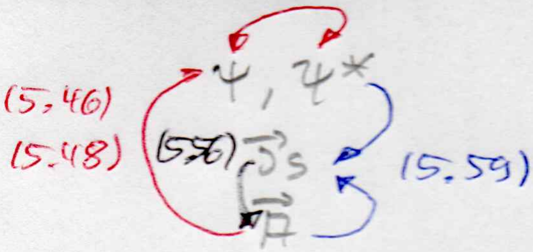
5.3.9 Summary:

Thus, we can summarize that the state of a superconductor within the Ginzburg-Landau theory is described by 8 degrees of freedom:

1) The statics is determined by ψ^*, ψ, which determine via (5.7) the density of superconducting electrons.
2) The stationary dynamics is covered by the superconducting electron current density $\vec{j}_s$.
3) The vector potential $\vec{A}$ fixes the magnetic induction $\vec{B}$ and, thus, the magnetic properties.

These 8 unknown components are determined by the volume conditions (5.46), (5.48), (5.59) and the Oersted law (5.56). Thus, these Ginzburg-Landau equations represent a complete, consistent set of equations for describing the state of an inhomogeneous superconductor.

The adjacent schematic figure illustrates the influences of the respective quantities upon each other.



5.3.10 Concluding Remarks:

An exact analytic solution of these Ginzburg-Landau equations is not possible due to the involved nonlinear couplings. Therefore, one needs to solve them numerically. In this lecture, however, we investigate solutions of the Ginzburg-Landau equations with the help of two approximative approaches:

1) Under certain conditions small terms can be neglected either a priori or a posteriori.
2) It is possible to assume under certain physical conditions that one of the quantities, for instance the vector potential $\vec{A}$, has a certain given spatial dependence. Then the other quantities can be determined under this assumption.

5.4 Characteristic Lengths:

The Ginzburg-Landau theory contains two characteristic length scales:

1) The London penetration length λ_L represents the length scale, within which the magnetic field can enter the superconductor.
2) The coherence length ξ represents the length scale upon which the order parameter Ψ and, thus, the superconducting electron density (5.7) vary.

In the following we discuss in more detail implications of the Ginzburg-Landau theory for the extreme cases of both type II and type I superconductors, where $\lambda_L \gg \xi$ and $\lambda_L \ll \xi$ hold, respectively.

5.4.1 Extreme Superconductor of Type II

The case $\lambda_L \gg \xi$ is fulfilled, for instance, by high T_c superconductors, which are typically characterised by $\lambda_L = 1500$ Å and $\xi = 10$ Å. The figure at the top of page 68 suggests in that case to neglect the gradient of the order parameter in the volume of the superconductor. As already mentioned at the end of Section 5.3.4, the case Ψ = const. and, thus, grad $\Psi = \vec{0}$ corresponds to the London theory. Thus, the London theory is a special case of the Ginzburg-Landau theory provided that $\lambda_L >> \xi$ holds.

Due to grad $\Psi^* =$ grad $\Psi = \vec{0}$ the superconducting current density (4.142) reduces to (4.143) with (5.7), which corresponds to the London theory. Taking the rotation of (4.143) with (5.7) then yields

$$\text{rot}\,\vec{j}_S = -\frac{e_S^2 n_S}{m_S}\,\text{rot}\,\vec{A} \overset{(4.30)}{=} -\frac{e_S^2 n_S}{m_S}\vec{B} \qquad (5.62)$$

which corresponds to the second London equation (4.29) due to (4.16). Inserting in (5.62) the Oersted law (5.56) we obtain

$$\text{rot rot}\,\vec{B} \overset{(4.40)}{=} \text{grad div}\,\vec{B} - \Delta\vec{B} \overset{(M2)}{=} -\Delta\vec{B} = -\frac{e_S^2 n_S \mu_0}{m_S}\vec{B} \qquad (5.63)$$

which coincides with the Helmholtz equation (4.52) due to (4.47). Thus from the point of view of the Ginzburg-Landau theory, the London penetration length (4.47) reads with (5.7)

$$\lambda_L = \sqrt{\frac{m_S}{e_S^2 \mu_0 |\Psi(T)|^2}} \qquad (5.64)$$

At the critical temperature $T = T_C$ the magnetic field is negligible, so the Ginzburg-Landau equation (5.46) reduces to the Landau equation (5.13) with the solution (5.15). According to (5.23) and (5.24) this guarantees that at $T \to T_C$ the transition from a superconductor to a normal conductor is characterized by a diverging London penetration depth, i.e. $\lambda_L \to \infty$.

5.4.2 Extreme Superconductor of Type I

In the case $\lambda_L << \xi$ we read off from the figure at the bottom of page 67 that the volume of an extreme type I superconductor is characterized by a vanishing magnetic induction, i.e. $\vec{B} = \vec{0}$. Thus, then the vector potential $\vec{A}$ can be neglected in the Ginzburg-Landau equations.

In this approximation the Ginzburg-Landau equation (5.46) reduces to the Gross-Pitaevskii equation

$$-\frac{\hbar^2}{2m_S}\Delta\Psi + \alpha\Psi + \beta|\Psi|^2\Psi = 0 \qquad (5.65)$$

In the one-dimensional real case (5.65) reduces to

$$-\frac{\hbar^2}{2m_S}\frac{d^2\psi}{dz^2}+\alpha\psi+\beta\psi^3=0 \qquad (5.66)$$

In order to solve (5.66) we introduce a dimensionless order parameter $\overline{\psi}$ by measuring ψ in units of the homogeneous superconductor of Section 5.2 according to (5.15)

$$\overline{\psi}=\frac{\psi}{\psi_0} \qquad (5.67)$$

Inserting (5.15) and (5.67) in (5.66) yields

$$-\frac{\hbar^2}{2m_S}\frac{d^2\overline{\psi}}{dz^2}+\alpha\overline{\psi}-\alpha\overline{\psi}^3=0 \qquad (5.68)$$

This suggests to define the length scale

$$\xi=\sqrt{-\frac{\hbar^2}{2m_S\alpha}} \qquad (5.69)$$

where we can assume $\alpha<0$ in the superconducting phase according to the Landau theory of Section 5.2. Thus, (5.68) reduces to

$$\xi^2\frac{d^2\overline{\psi}}{dz^2}+\overline{\psi}-\overline{\psi}^3=0 \qquad (5.70)$$

Introducing the dimensionless coordinate $\overline{z}$ by measuring z in units of the London penetration length λ_L according to

$$\overline{z}=\frac{z}{\lambda_L} \qquad (5.71)$$

then yields

$$\frac{1}{\varkappa^2}\frac{d^2\overline{\psi}}{d\overline{z}^2}+\overline{\psi}-\overline{\psi}^3=0 \qquad (5.72)$$

with the Ginzburg-Landau parameter (4.69). Multiplying (5.72) with $d\overline{\psi}/d\overline{z}$ allows to integrate the second-order differential equation once:

$$\frac{1}{2\varkappa^2}\left(\frac{d\overline{\psi}}{d\overline{z}}\right)^2+\frac{1}{2}\overline{\psi}^2-\frac{1}{4}\overline{\psi}^4+c_1=0 \qquad (5.73)$$

How can we fix the integration constant c_1? In the limit $\overline{z}\to\infty$, i.e. deep in the superconductor, we have to guarantee $\overline{\psi}\to 1$ and $d\overline{\psi}/d\overline{z}\to 0$. With this we conclude

$$\frac{1}{2}-\frac{1}{4}+c_1=0 \Rightarrow c_1=-\frac{1}{4} \qquad (5.74)$$

Inserting (5.74) in (5.73) yields

$$\frac{1}{\varkappa^2}\left(\frac{d\overline{\psi}}{d\overline{z}}\right)^2=\frac{1}{2}-\overline{\psi}^2+\frac{1}{2}\overline{\psi}^4=\frac{1}{2}\left(1-\overline{\psi}^2\right)^2 \qquad (5.75)$$

The method of separating variables then gives due to $0 \leq \overline{\Psi} < 1$:

$$\frac{d\overline{\Psi}}{1-\overline{\Psi}^2} = \frac{\varkappa}{\sqrt{2}}\, d\overline{z} \qquad (5.76)$$

With the stem functions on both sides we get

$$\operatorname{artanh} \overline{\Psi} = \frac{\varkappa}{\sqrt{2}} \overline{z} + c_2 \Rightarrow \overline{\Psi}(\overline{z}) = \tanh\left(\frac{\varkappa}{\sqrt{2}} \overline{z} + c_2\right) \qquad (5.77)$$

But (5.77) can only be a solution for the order parameter within the superconductor, i.e. we have to restrict (5.77) to the half axis $\overline{z} \geq 0$. Conversely, for $\overline{z} \leq 0$, i.e. in the normal conductor, the order parameter must vanish

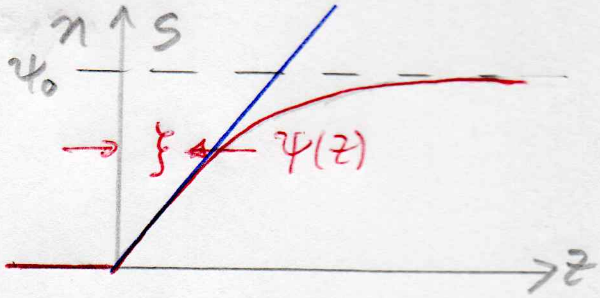
$$\overline{\Psi}(\overline{z}) = 0 \qquad (5.78)$$

which is, indeed, a trivial solution of (5.72). As the solutions (5.77) and (5.78) for $\overline{z} \geq 0$ and $\overline{z} \leq 0$ have to be continuous at $\overline{z} = 0$, we determine the second integration constant c_2 to be given by $c_2 = 0$ and, finally, obtain

$$\overline{\Psi}(\overline{z}) = \begin{cases} \tanh\left(\frac{\varkappa}{\sqrt{2}} \overline{z}\right) & ; z \geq 0 \\ 0 & ; z \leq 0 \end{cases} \qquad (5.79)$$

Restoring the substitutions (5.70), (5.71) yields

$$\Psi(z) = \begin{cases} \Psi_0 \tanh \frac{z}{\sqrt{2}\,\xi} & ; z \geq 0 \\ 0 & ; z \leq 0 \end{cases} \qquad (5.80)$$



It is remarkable that the order parameter Ψ increases linearly at the surface of the superconductor.

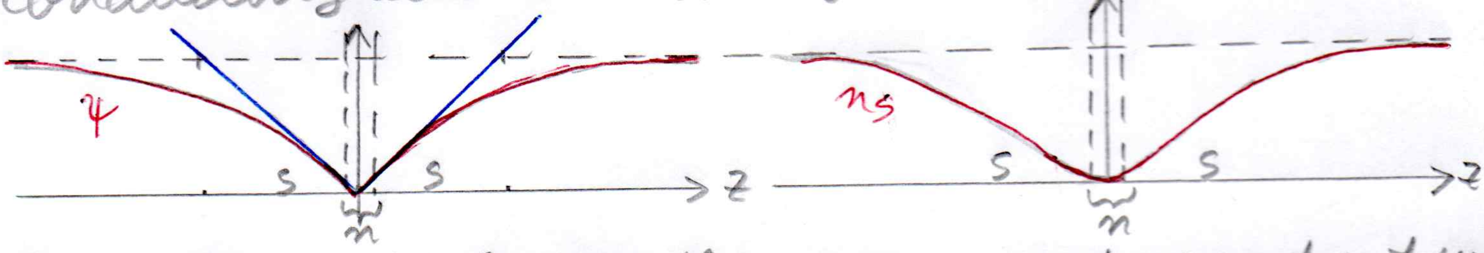
From (5.80) we read off that the order parameter Ψ varies on the order of the coherence length. From (5.10) and (5.69) we read off that the coherence length diverges at the critical point $T = T_c$ according to

$$\xi \sim (T_c - T)^{-0.5} \qquad (5.81)$$

Thus, due to (5.24) and (5.81) both characteristic length scales of the Ginzburg-Landau theory diverge at the critical point with the same critical exponent 0.5.

Furthermore, we can use the result (5.80) for a useful qualitative conclusion concerning a flux quantum.

As a model we can consider a flux quantum as an sns-transition, where the intermediate n region shrinks to zero. With this we obtain the following spatial dependences for both the order parameter Ψ and the superconducting electron density n_s:



Thus, whereas the London theory assumed a constant superconducting electron density, the Ginzburg-Landau theory suggests that n_s decreases quadratically if one goes from the center of the flux quantum outwards.