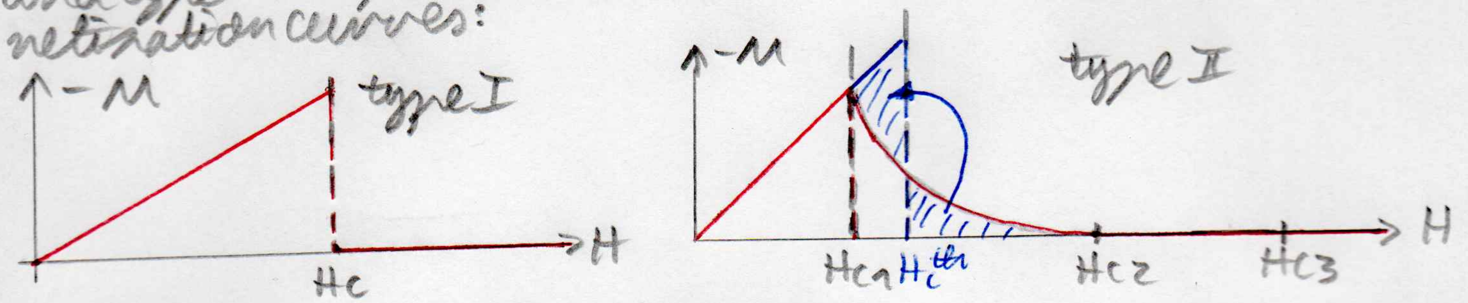
6 Critical Fields:

In this chapter we discuss in detail how the Ginzburg-Landau theory allows to determine the critical magnetic fields for both type I and type II superconductors. This represents an interesting application of the Ginzburg-Landau theory, which is of practical use.

In order to simplify the considerations we assume that the superconducting probe has the shape of a long stretched cylinder. According to the discussion in subsection 2.4.2 it is then justified to neglect effects originating from any stray field as the corresponding demagnetisation factor vanishes.

6.1 Overview:

Let us briefly summarize what we know so far about the critical magnetic fields. According to chapter 2 type I and type II superconductors differ in their respective magnetisation curves:



H_{c1}: The Meissner-Ochsenfeld effect, that the magnetic induction is completely expelled from the superconductor, starts to become no longer valid. To this end the first flux quanta enter the volume of the superconductor. The center of the flux quanta are normal conducting, so the magnetic induction is non-vanishing there.

H_{c2}: The whole superconductor consists of flux quanta. But superconductivity still persists at the surface.

H_{c3}: Also the surface superconductivity breaks down.

For superconductors of type II it is possible to define a thermodynamic critical magnetic field H_c^{th} as follows. To this end one applies a Maxwell construction, which generically occurs for phase transitions of first order. Namely, one considers the magnetisation energy of the type II superconductor

$$E^{II}_{mag} = -\mu_0 V \int_0^{H_{c2}} dH \, M(H) \tag{6.1}$$

which corresponds to the area below the magnetisation curve, and identifies it with the magnetisation energy of a fictitious type I superconductor

$$E^{I}_{mag} = \frac{\mu_0}{2} V \, H_c^{th\,2} \qquad (6.2)$$

This yields a condition defining the thermodynamic critical field H_c^{th}:

$$-\int_0^{H_{c2}} dH\, M(H) = \frac{1}{2} H_c^{th\,2} \qquad (6.3)$$

Note a similar energetic consideration was performed for type I superconductors for their intermediate state in (2.38).

In the following we apply different concepts in order to determine the critical magnetic fields within the realm of the Ginzburg-Landau theory:

1) The upper critical field H_{c2} represents a volume field strength, where the superconducting order parameter starts to become non-zero. Therefore, it will turn out that H_{c2} follows from linearising the Ginzburg-Landau equations and from solving the corresponding eigenvalue problem.

2) Instead, the lower critical field H_{c1} follows from thermodynamic considerations. To this end one equates the free enthalpy of the Meissner and the Shubnikov phase

$$G_{Meissner}(H_{c1}) = G_{Shubnikov}(H_{c1}) \qquad (6.4)$$

and determines from that condition H_{c1}.

6.2 Upper Critical Field H_{c2}:

The state of a superconductor is described by the fields Ψ^*, Ψ, $\vec{A}$ and $\vec{j}_s$. According to the Ginzburg-Landau theory they are coupled by the equations (5.46), (5.56) and (5.59):

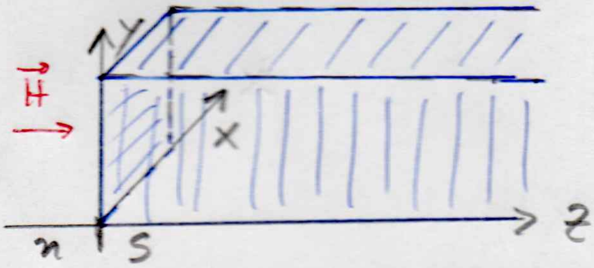
$$\frac{1}{2m_s}\left(-i\hbar\vec{\nabla} - e_s\vec{A}\right)^2\Psi = -\alpha\Psi - \beta|\Psi|^2\Psi \qquad (6.5)$$

$$\vec{j}_s = \frac{i\hbar e_s}{2m_s}\left(\Psi\vec{\nabla}\Psi^* - \Psi^*\vec{\nabla}\Psi\right) - \frac{e_s^2}{m_s}\vec{A}|\Psi|^2 \qquad (6.6)$$

$$\text{rot}\,\vec{B} = \mu_0\,\vec{j}_s \qquad (6.7)$$

Thus, the question arises which assumptions allow to determine the upper critical field H_{c2} from these coupled equations. To this end, we have to specify in particular the underlying geometry.

6.2.1 Assumptions:



To be specific, let us choose as a geometry the adjacent figure. The xy-plane separates the right half-space of the superconductor from the left half-space of the normal conductor or the vacuum. The external magnetic field $\vec{H}$ is supposed to be pointing in z-direction, so it is perpendicular to the ns-interface. This geometry allows now for the following simplifications to determine H_{c2}.

Coming from higher magnetic fields we investigate the transition point H_{c2}, where the superconductor just starts to turn from the normal conducting into the superconducting state. Thus, only few superconducting electrons exist and the order parameter Ψ is small in comparison to the bulk equilibrium value Ψ_0:

$$\Psi \ll \Psi_0 \tag{6.8}$$

The smallness condition (6.8) allows to linearise the Ginzburg-Landau equations (6.5)-(6.7) with respect to the order parameter Ψ. At first, we obtain due to the linearization in Ψ from (6.6) that there are no superconducting currents in the bulk:

$$\vec{j}_s(\vec{r}) = \vec{0} \tag{6.9}$$

According to (6.7) this has the consequence that the magnetic induction $\vec{B}$ turns out to be spatially homogeneous in the bulk and is given by the external magnetic field:

$$\vec{B}(\vec{r}) = \mu_0 \vec{H}(\vec{r}) = \mu_0 H_z \vec{e}_z = B_z \vec{e}_z \tag{6.10}$$

A vector potential $\vec{A}(\vec{r})$, which respects the Coulomb gauge (4.129), and corresponds to the magnetic induction (6.10) due to (4.30) is given by

$$\vec{A}(\vec{r}) = B_z \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} \tag{6.11}$$

Indeed, we get

$$\operatorname{rot} \vec{B}(\vec{r}) \underset{=}{\overset{(4.30),(6.11)}{=}} \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & B_z x & 0 \end{vmatrix} = B_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \hat{=} (6.10)$$

And, finally, a linearization of the Ginzburg-Landau equation (6.5) with respect to Ψ yields the equation

$$\frac{1}{2m_S}\left(-i\hbar\vec{\nabla}-e_S\vec{A}\right)^2\Psi=-\alpha\Psi \qquad (6.12)$$

which represents an eigenvalue problem.

6.2.2 Eigenvalue Problem:

Multiplying out the brackets, (6.12) reduces to

$$-\frac{\hbar^2}{2m_S}\Delta\Psi+i\frac{\hbar e_S}{m_S}\vec{A}\cdot\vec{\nabla}\Psi+\frac{i\hbar e_S}{2m_S}(\vec{\nabla}\vec{A})\Psi+\frac{e_S^2}{2m_S}\vec{A}^2\Psi=-\alpha\Psi \qquad (6.13)$$

Inserting therein the vector potential (6.11) yields

$$-\frac{\hbar^2}{2m_S}\Delta\Psi+\frac{i\hbar e_S}{m_S}B_Z x\frac{\partial\Psi}{\partial y}+\frac{e_S^2}{2m_S}B_Z^2x^2\Psi=-\alpha\Psi \qquad (6.14)$$

Here it is reasonable to perform the separation ansatz

$$\Psi(x,y,z)=e^{i(\gamma y+\delta z)}\varphi(x) \qquad (6.15)$$

which effectively reduces the three-dimensional problem to a one-dimensional one:

$$-\frac{\hbar^2}{2m_S}\frac{d^2\varphi(x)}{dx^2}-\frac{\hbar e_S}{m_S}B_Z\gamma x\varphi(x)+\frac{e_S^2B_Z^2}{2m_S}x^2\varphi(x)=-\left[\alpha+\frac{\hbar^2}{2m_S}(\gamma^2+\delta^2)\right] \qquad (6.16)$$

A quadratic completion allows to convert (6.16) to the standard form of a Schrödinger equation for a one-dimensional harmonic oscillator:

$$\frac{e_S^2B_Z^2}{2m_S}x^2-\frac{\hbar e_S}{m_S}B_Z\gamma x=\frac{e_S^2B_Z^2}{2m_S}\left(x-\frac{\hbar\gamma}{e_SB_Z}\right)^2-\frac{\hbar^2\gamma^2}{2m_S} \qquad (6.17)$$

Due to (6.17) equation (6.16) can be recast into

$$-\frac{\hbar^2}{2m_S}\frac{d^2\varphi(x)}{dx^2}+\frac{e_S^2B_Z^2}{2m_S}\left(x-\frac{\hbar\gamma}{e_SB_Z}\right)^2\varphi(x)=-\left(\alpha+\frac{\hbar^2\delta^2}{2m_S}\right) \qquad (6.18)$$

This form suggests to perform the translation

$$x'=x-\frac{\hbar\gamma}{e_SB_Z}, \qquad (6.19)$$

to introduce the cyclotron frequency

$$\omega_0=\frac{e_SB_Z}{m_S} \qquad (6.20)$$

and to redefine

$$\alpha'=\alpha+\frac{\hbar^2\delta^2}{2m_S} \qquad (6.21)$$

Indeed, with applying (6.19)–(6.21), we can convert (6.18) into the standard Schrödinger equation for a one-dimensional harmonic oscillator:

$$\left(-\frac{\hbar^2}{2m_S}\frac{d^2}{dx'^2}+\frac{1}{2}m_S\omega_0^2x'^2\right)\varphi(x')=-\alpha'\varphi(x') \qquad (6.22)$$

Due to quantum mechanics we obtain immediately the solutions of this eigenvalue problem. The eigenvalues read

$$-\alpha'_n = \hbar\omega_0\left(n+\frac{1}{2}\right) \tag{6.23}$$

and the eigenfunctions are given by

$$\varphi_n(x') = N_n H_n\left(\sqrt{\frac{m_S\omega_0}{\hbar}}\,x'\right) e^{-\frac{m_S\omega_0}{\hbar}x'^2} \tag{6.24}$$

6.2.3 Quantum Mechanics:

In quantum mechanics the motion of a charged particle in a homogeneous magnetic field is described by the eigenvalue problem (6.22). The allowed energy levels follow from (6.21) and (6.23):

$$E_n = -\alpha_n = \hbar\omega_0\left(n+\frac{1}{2}\right) + \frac{\hbar^2\delta^2}{2m_S} \tag{6.25}$$

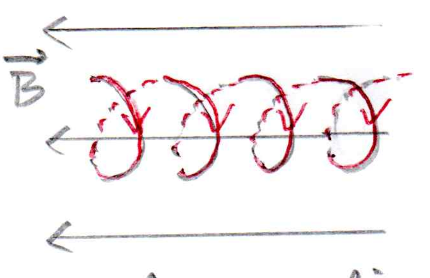
The second term corresponds to the kinetic energy of the translational motion in z-direction, whereas the first term describes the energy of the motion in the perpendicular plane. Note that the latter energy can be recast with taking into account (6.20) into the potential energy of a magnetic moment in the magnetic field (6.10):

$$E_n = -\vec{m}_n\cdot\vec{B} + \frac{\hbar^2\delta^2}{2m_S}, \quad m_{zn} = m_B(2n+1) \tag{6.26}$$

Here the Bohr magneton

$$m_B = -\frac{\hbar e_S}{2m_S} \tag{6.27}$$

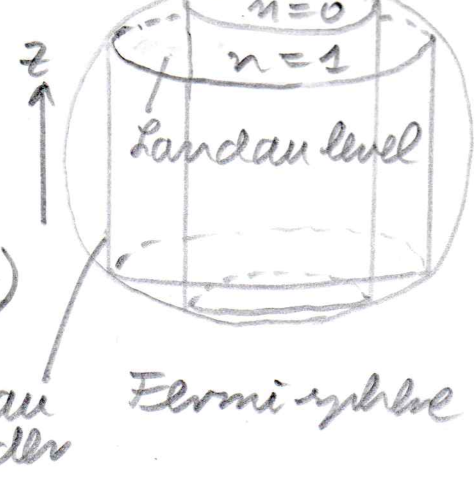
appears, which represents the unit of the magnetic moment of the charge $-e_S$ and the mass m_S. Furthermore, the eigenfunctions (6.24) correspond to the classical circular motion of a charged particle in the xy-plane, which is perpendicular to the direction of the magnetic field:



circular motion of charge $-e_S$ with cyclotron frequency ω_0

But the allowed energy levels are discrete cylinder tubes:

$$\frac{\hbar^2}{2m_S}\left(k_x^2+k_y^2\right) = \omega_0\left(n+\frac{1}{2}\right) \tag{6.28}$$



6.2.4 Upper Critical Field:

In quantum mechanics the magnetic field is applied from outside and, therefore, is given. And one determines from the eigenvalue problem the allowed Landau levels as the energy eigenvalues.

But in the Ginzburg – Landau theory of a superconductor the inverse problem occurs. Here the eigenvalue α represents a Landau coefficient, whose temperature dependence is a phenomenologically given according to (5.10). Thus, conversely, we are aiming here for the largest possible magnetic field as this defines the upper critical field B_{c2}.

To this end we use (6.20), (6.20), and (6.23) in order to solve for the magnetic field:

$$B_z(n,\delta) = \frac{2m_s}{\hbar e_s}\left(-\alpha - \frac{\hbar^2\delta^2}{2m_s}\right)\frac{1}{2n+1} \tag{6.29}$$

Note that for $T < T_c$ we have $\alpha < 0$ due to (5.10). We read off from (6.29) that the magnetic field $B_z(n,\delta)$ is largest for the quantum numbers $n=0$ and $\delta=0$:

$$B_{c2} = \max_{\substack{n\in\mathbb{N}\\ \delta\in\mathbb{R}}} B_z(n,\delta) = B_z(0,0) \overset{(6.29)}{=} \frac{2m_s}{\hbar e_s}(-\alpha) \tag{6.30}$$

All other magnetic fields $B_z(n,\delta)$ for $n \neq 0$ and $\delta \neq 0$ do not have any physical meaning as the approximation (6.9) would no longer hold.

6.2.5 Consequences:

We can now relate the upper critical field (6.30) to the thermodynamic critical field B_c^{th}. The latter follows from identifying the magnetic field energy (6.2) with the condensation energy (5.17) of the Landau theory:

$$\frac{\alpha^2}{2\beta} = \frac{B_c^{th\,2}}{2\mu_0} \quad\Rightarrow\quad B_c^{th} = \sqrt{\frac{\mu_0}{\beta}}\,(-\alpha) \tag{6.31}$$

The ratio of the magnetic fields (6.30) and (6.31) turns out to be temperature independent due to (5.11):

$$\frac{B_{c2}}{B_c^{th}} = \sqrt{\frac{\beta}{\mu_0}}\cdot\frac{2m_s}{\hbar e_s} \tag{6.32}$$

In order to interpret (6.32) physically, we have to recall that the Ginzburg – Landau theory is charac–

terized by two length scales, namely the London penetration length

$$\lambda_L \overset{(5.23),(5.64)}{=\!=} \sqrt{\frac{m_s\beta}{e_s^2\mu_0\alpha_c'(T_c-T)}} \sim \frac{1}{\sqrt{T_c-T}} \qquad (6.33)$$

and the coherence length

$$\xi \overset{(5.10),(5.69)}{=\!=} \sqrt{\frac{\hbar^2}{2m_s(T_c-T)\alpha_c'}} \sim \frac{1}{\sqrt{T_c-T}} \qquad (6.34)$$

whose ratio yields the temperature independent Ginzburg-Landau parameter

$$\varkappa \overset{(5.4)}{=\!=} \frac{\lambda_L}{\xi} \overset{(6.33),(6.34)}{=\!=} \sqrt{\frac{2\beta}{\mu_0}} \cdot \frac{m_s}{\hbar e_s} \qquad (6.35)$$

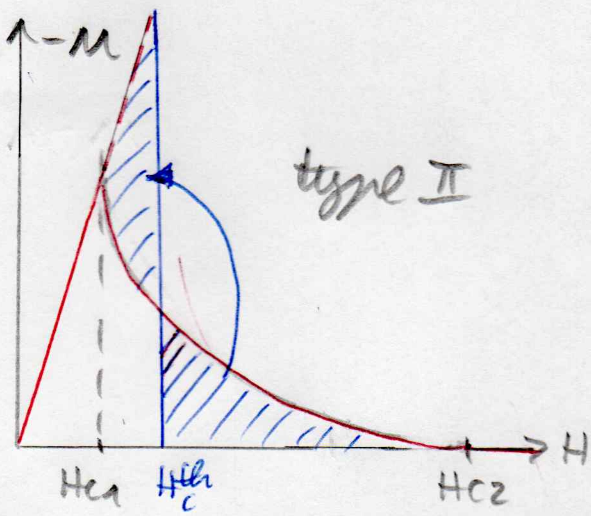
Thus, the ratio of the magnetic fields (6.32) is given by the dimensionless Ginzburg-Landau parameter:

$$\frac{B_{c2}}{B_c^{th}} = \sqrt{2}\,\varkappa \quad \Rightarrow \quad B_{c2} = \sqrt{2}\,\varkappa B_c^{th} \qquad (6.36)$$

6.2.6 Discussion:

1) Some exemplary values for the Ginzburg-Landau parameter

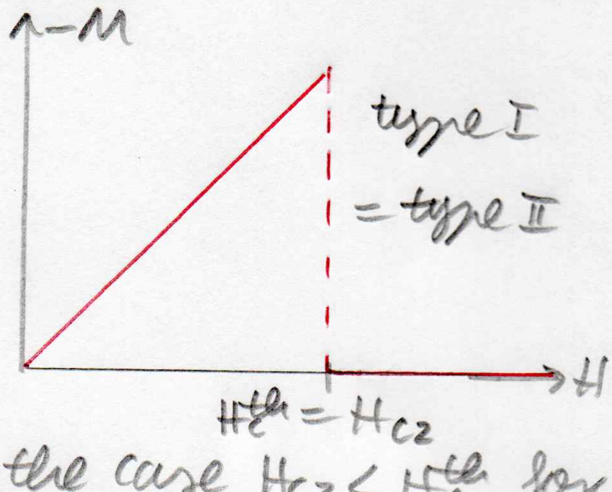
material	T_c	$\varkappa$
Nb_3Sn	18K	30
$YBa_2CuO_3O_7$	93K	100



show that superconductors of type II have, due to (6.36), the generic property

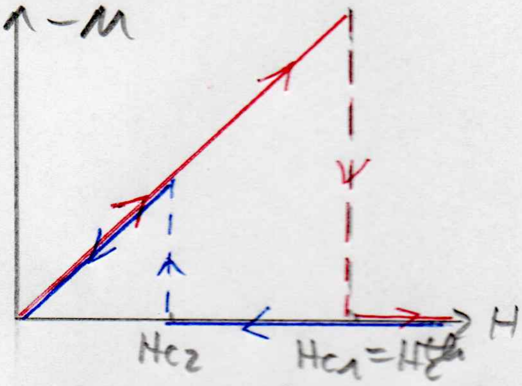
$$B_{c2} \gg B_c^{th} \qquad (6.37)$$

2) From (6.36) we also read off that at the limiting case $\varkappa = 1/\sqrt{2}$ we obtain $B_{c2} = B_c^{th}$. Thus, the magnetization curve of type II and type I superconductors turn out to coincide.



3) Which physical meaning has the case $H_{c2} < H_c^{th}$ for superconductors of type I, which are characterized

by $\varkappa < \varkappa_c = 1/\sqrt{2}$? Here one observes a hysteresis effect. Coming from higher magnetic fields the Meissner phase starts at $H_{c2} < H_c^{th}$, whereas coming from lower magnetic fields the Meissner phase ends at $H_{c1} = H_c^{th}$. If one measures the hysteresis, i.e. the two magnetic fields H_{c2} and H_c^{th}, then one can determine due to (6.36) the upper boundary of the Ginzburg-Landau parameter, i.e. $\varkappa_c = 1/\sqrt{2} \approx 0.7$. For instance, for pure aluminium one gets $\varkappa = 0.03$ and for pure indium we have $\varkappa = 0.06$.



Note that a similar hysteresis effect is well known at the first-order phase transition from liquid to solid water and is called there hypothermia ("Unterkühlung"). This hysteresis effect therefore underlines the previous statement that the transition from a normal to a superconducting phase at non-vanishing magnetic field is of first order.

4) According to (5.10) the upper critical field (6.32) has the temperature dependence

$$B_{c2} \sim (T_c - T)^{\gamma} \qquad (6.37)$$

with the mean-field exponent $\gamma_{MF} = 1$. Instead, experiments yield the value $\gamma = 0.64$. This discrepancy can be explained by the huge thermal fluctuations, which are present in the vicinity of the critical point $T = T_c$.

5) From the magnetic flux quantum (4.86) and the coherence length (6.34) we conclude

$$\frac{\phi_0}{2\pi\xi^2} = \frac{\hbar 2\pi}{e_s} \frac{2m_s(T_c - T)\alpha'}{2\pi\hbar^2} \overset{(5.10)}{=} \frac{2m_s}{\hbar e_s}(-\alpha) \overset{(6.30)}{=} B_{c2} \qquad (6.38)$$

Thus, at the upper critical field B_{c2} one single flux quantum ϕ_0 is frozen in the area $2\pi\xi^2$.

6) Combining (6.36) and (6.38) we conclude for the thermodynamic critical field

$$B_c^{th} \overset{(6.36)}{=} \frac{B_{c2}}{\sqrt{2}\,\varkappa} = \frac{\phi_0}{\sqrt{2}\, 2\pi\xi^2\varkappa} \overset{(5.4)}{=} \frac{\phi_0}{2\sqrt{2}\,\pi\lambda_L\xi} \qquad (6.39)$$

Thus, B_c^{th} corresponds to a flux quantum ϕ_0 being frozen in the area $2\sqrt{2}\,\pi\lambda_L\xi$.

6.2.7 Flux Line Lattice:

Due to the upper critical field (6.30) only the eigenfunction (6.24) with $n=0$ appears. It represents a Gauss function with a width, which is given by the oscillator length

$$l_{osc} = \sqrt{\frac{\hbar}{m_s \omega_0}} \qquad (6.40)$$

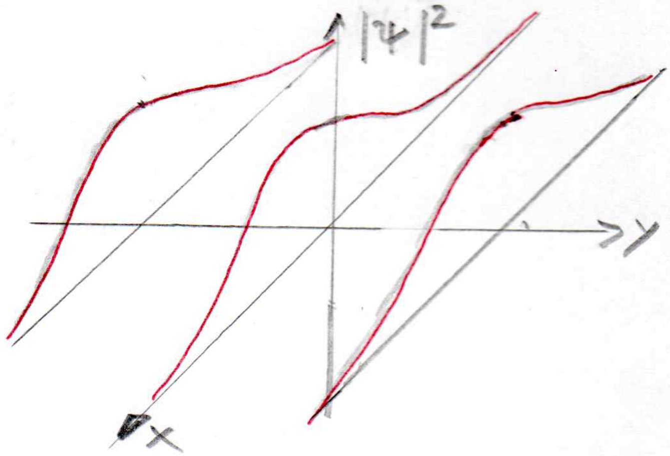
Taking into account the cyclotron frequency (6.20), the oscillator length (6.40) turns out to be independent of the mass m_s:

$$l_{osc} \overset{(6.20),(6.40)}{=} \sqrt{\frac{\hbar}{e_s B_z}} \qquad (6.41)$$

Evaluating (6.41) at the upper critical field (6.30) yields the result that the oscillator length coincides with the coherence length

$$l_{osc} \overset{(6.30),(6.41)}{=} \sqrt{\frac{\hbar^2}{2 m_s (T_c - T)\alpha_c'}} \overset{(6.34)}{=} \xi \qquad (6.42)$$

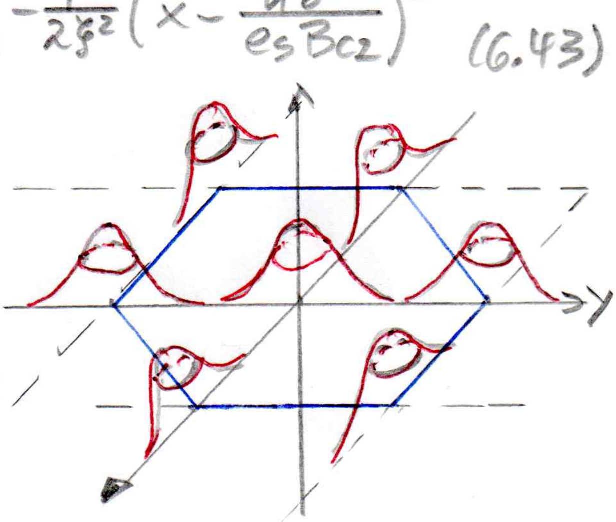
Furthermore, we note that the eigenfunctions following from (6.15) and (6.24) for $n=0$ do not yield any density modulation in y-direction. Therefore, they are not suitable to describe a flux line lattice, which is supposed to exist in the Shubnikov phase.



But in the separation ansatz (6.15), where we have to set $\delta=0$ due to the upper critical field (6.30), we still have a degeneracy with respect to γ. Thus, combining (6.15), (6.19), (6.24), (6.42) the general solution is given by the superposition

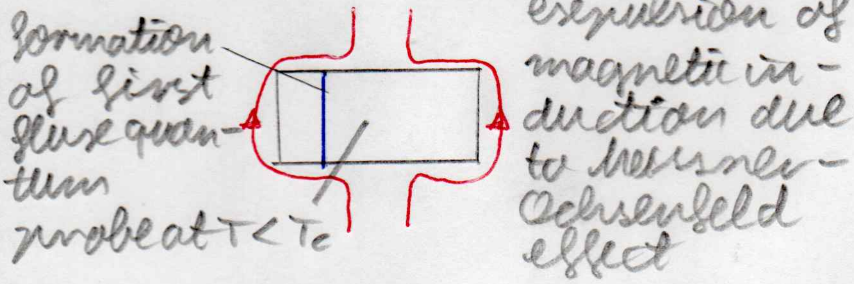
$$\Psi(x,y,z) = \int d\gamma \, c(\gamma) \, e^{i\gamma y} \, e^{-\frac{1}{2\xi^2}\left(x - \frac{\hbar\gamma}{e_s B_{c2}}\right)^2} \qquad (6.43)$$

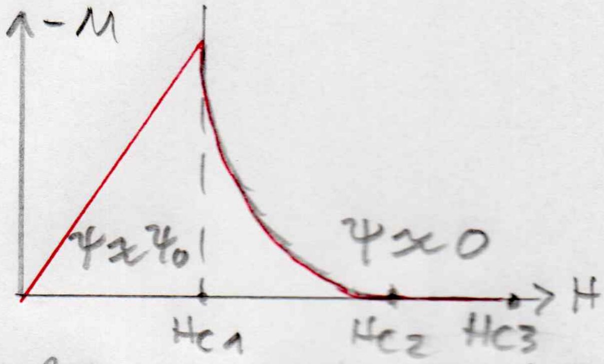
A suitable choice of $c(\gamma)$ yields now a modulation of the order parameter in the xy-plane. Candidates for a two-dimensional flux line lattice are a square and a hexagonal lattice. The former was predicted by Abrikosov, but the experiment shows that the latter is realised in the Shubnikov phase.



6.4 Lower Critical Field H_{c1}:

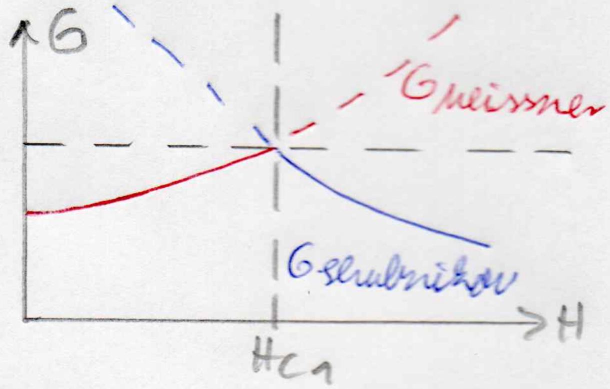
The lower critical field H_{c1} is characterised by the formation of the first flux quantum:





The transition from the Meissner phase $0 \leq H \leq H_{c1}$, where no flux quanta exist, to the Shubnikov phase $H_{c1} \leq H \leq H_{c2}$, where flux quanta do exist, can be described as a phase transition from the point of view of thermodynamics. The lower critical field H_{c1} turns out to result from equating the respective free enthalpies for both phases according to (6.4).

Whereas G_{Meissner} is lower than $G_{\text{Shubnikov}}$ for $0 \leq H \leq H_{c1}$, conversely, $G_{\text{Shubnikov}}$ is lower than G_{Meissner} for $H_{c1} \leq H \leq H_{c2}$. At $H = H_{c1}$ both free enthalpies coincide, so (6.4) determines H_{c1}.



6.3.1 Starting Point:

We assume the superconductor to be so large, that surface effects are negligible. This means, in particular, that surface terms can be neglected when the free enthalpies in both the Meissner and the Shubnikov phase are evaluated. According to the Ginzburg-Landau theory the full expression for the free enthalpy is given by (5.39).

6.3.2 Meissner Phase:

In the Meissner phase a flux quantum has not yet emerged. Thus, spatial inhomogeneities of the order parameter Ψ can only occur at the surface of the superconductor. But such surface contributions can be neglected according to the previous subsection. Therefore, we specialize (5.39) in the Meissner phase by demanding

$$\Psi = \Psi_0, \quad \vec{\nabla}\Psi = \vec{\nabla}\Psi^* = \vec{0} \tag{6.44}$$

Furthermore, we know that the Meissner-Ochsenfeld effect is present in the Meissner phase, so we have also

$$\vec{B} = \vec{H} = \vec{0} \qquad (6.45)$$

Inserting (6.44) and (6.45) in (5.39) we obtain for the free enthalpy of the Meissner phase:

$$G_{Meissner} = G_n + \int_V dV \left\{ \alpha |\Psi_0|^2 + \frac{\beta}{2} |\Psi_0|^4 + \frac{\mu_0}{2} \vec{H}^2 \right\} \qquad (6.46)$$

6.3.3 Shubnikov Phase:

In the Shubnikov phase we can not make any simplifying assumption as (6.44) and (6.45). As one flux quantum exist, the order parameter changes there spatially and also the Meissner-Ochsenfeld effect breaks down. Thus, in the Shubnikov phase the full expression of the free enthalpy (5.39) has to be taken into account.

6.3.4 Free Enthalpy Balance:

Inserting (5.39) for the Shubnikov phase and (6.46) for the Meissner phase into (6.4) we obtain the following equilibrium condition for determining the lower critical field:

$$0 = \int_V dV \Big\{ \underbrace{\alpha\left[|\Psi|^2 - |\Psi_0|^2\right] + \frac{\beta}{2}\left[|\Psi|^4 - |\Psi_0|^4\right] + \frac{1}{2m_s}\left|-i\hbar\vec{\nabla}\Psi - e_s\vec{A}\Psi\right|^2 + \frac{\vec{B}^2}{2\mu_0}}_{=①} \underbrace{- \vec{B}\vec{H}_{c_1}}_{=②} \Big\} \qquad (6.47)$$

Both highlighted contributions allow for a concrete physical interpretation:

① The first term is only non-vanishing in the vicinity of a flux line. Assuming a cylinder symmetry for the flux quantum, the volume integral can be decomposed as follows

$$dV = dF \cdot L \qquad (6.48)$$

With this we obtain the line energy ε_L with the physical dimension $[\varepsilon_L] = 1\,\mathrm{J/m}$, which is needed for generating the flux quantum:

$$\varepsilon_L = \int dF \left\{ \alpha\left[|\Psi|^2 - |\Psi_0|^2\right] + \frac{\beta}{2}\left[|\Psi|^4 - |\Psi_0|^4\right] + \frac{1}{2m_s}\left|-i\hbar\vec{\nabla}\Psi - e_s\vec{A}\Psi\right|^2 + \frac{\vec{B}^2}{2\mu_0} \right\} \qquad (6.49)$$

② The second term represents the interaction of the flux quantum with the magnetic field. It corresponds to the energy, which is released due to the generation of the flux quantum. Applying the decomposition (6.48) also to the second term, the equilibrium condition (6.47) is rewritten as

$$\underbrace{\varepsilon_L}_{\text{energy cost per length}} - \underbrace{H_{c1} \cdot \int_F B\, dF}_{\text{energy gain per length}} = 0 \qquad (6.50)$$

In case that only only one single flux quantum is present, we have

$$\int_F B\, dF = \phi_0 \qquad (6.51)$$

so the lower critical field H_{c1} is given by

$$H_{c1} = \frac{\varepsilon_L}{\phi_0} \qquad (6.52)$$

Thus, the calculation of the lower critical field H_{c1} is traced back to determining the line energy (6.49).

6.3.5 Splitting of Line Energy:

For practical purposes we split the line energy (6.49) for a single flux quantum into two pieces:

$$\varepsilon_L = \varepsilon_L^{\psi} + \varepsilon_L^{B} \qquad (6.53)$$

The first piece ε_L^{ψ} depends on the order parameter ψ, whereas the second piece ε_L^{B} is governed by the magnetic field. Note that the interaction energy "$-\vec{j}_s \cdot \vec{A}$" from (5.34) between the superconducting current density $\vec{j}_s$ and the vector potential $\vec{A}$ is partially counted for ε_L^{ψ} and ε_L^{B}, respectively, due to (5.59):

$$\varepsilon_L^{\psi} = \int dF \left\{ \alpha\left[|\psi|^2 - |\psi_0|^2\right] + \frac{\beta}{2}\left[|\psi|^4 - |\psi_0|^4\right] + \frac{\hbar^2}{2m_s}|\vec{\nabla}\psi|^2 \underbrace{- \frac{i\hbar e_s}{2m_s}\left(\psi \vec{\nabla}\psi^* - \psi^* \vec{\nabla}\psi\right)}_{\text{first current term}} \right\} \qquad (6.54)$$

$$\varepsilon_L^{B} = \int dF \left\{ \frac{\vec{B}^2}{2\mu_0} + \underbrace{\frac{e_s^2}{2m_s} \vec{A}^2 \cdot |\psi|^2}_{\text{second current term}} \right\} \qquad (6.55)$$

An exact calculation of the line energy (6.53)–(6.55) would assume to know the precise spatial shape of both the order parameter ψ and the vector potential $\vec{A}$ in the vicinity of the flux quantum. To this end one would have to solve for this situation the corresponding Ginzburg-Landau equations. As this is not possible, we reside here to a specific approximation.

6.3.6 Approximation:

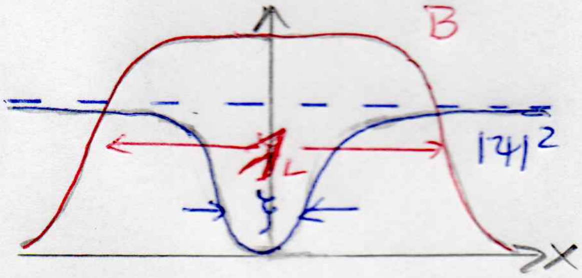
In the following we proceed with the calculation of the line energy (6.53)–(6.55) for the case

$$\varkappa = \frac{\lambda_L}{\xi} \gg 1 \qquad (6.55)$$

i.e. for extreme type II superconductors. This is insofar reasonable as the Shubnikov phase only exists for type II superconductors. For instance, the assumption (6.55) is well justified for high-T_c superconductors, where we have about $\varkappa = 100$. And, finally, we note that (6.55) corresponds to the situation that the Ginzburg-Landau theory goes over into the London theory as was already discussed in Subsection 5.4.1. Thus, in the following we only have to find an efficient way how to implement the approximation (6.55) into the concrete calculation.

6.3.7 Calculation of $\mathcal{E}^{\Psi}_{L}$:

The adjacent figure suggests to ignore the gradient terms of the order parameter in (6.54), see the corresponding discussion in Subsection 5.4.1.



Although Subsection 5.4.2 dealt with the case opposite to (6.55), i.e. $\lambda_L \ll \xi$, we can still use here the generic result that the order parameter varies linearly in the center of the flux quantum. Thus, we perform the ansatz

$$\Psi(r) = \Psi_0 \frac{r}{\xi} e^{i\varphi}, \quad \Psi_0^2 = -\frac{\alpha}{\beta}, \quad 0 \le r \le \xi \qquad (6.56)$$

Due to this ansatz (6.56) the first current term in (6.54) vanishes and we have only a contribution stemming from Ψ

$$\mathcal{E}^{\Psi}_{L,1} = \int dF \left\{ \alpha \left[|\Psi|^2 - |\Psi_0|^2 \right] + \frac{\beta}{2} \left[|\Psi|^4 - |\Psi_0|^4 \right] \right\} \qquad (6.57)$$

and another one depending on grad Ψ

$$\mathcal{E}^{\Psi}_{L,2} = \int dF \, \frac{\hbar^2}{2m} \left| \vec{\nabla} \Psi \right|^2 \qquad (6.58)$$

Inserting (5.56) in (6.57) yields

$$\mathcal{E}^{\Psi}_{L,1} = \int dF \left\{ \alpha \frac{-\alpha}{\beta} \left(\frac{r^2}{\xi^2} - 1 \right) + \frac{\beta}{2} \frac{\alpha^2}{\beta^2} \left(\frac{r^4}{\xi^4} - 1 \right) \right\} = \int dF \frac{\alpha^2}{2\beta} \left(1 - 2\frac{r^2}{\xi^2} + \frac{r^4}{\xi^4} \right) \qquad (6.59)$$

Taking into account (6.31), i.e. the condensation energy expressed in terms of the thermodynamic critical field B_c^{th}, and evaluating (6.59) further with polar coordinates, we get

$$\varepsilon^{\psi}_{L,1} = \frac{B_c^{th\,2}}{2\mu_0} 2\pi \int_0^{\xi} dr \left(r - 2\frac{r^3}{\xi^2} + \frac{r^5}{\xi^4} \right) = \frac{B_c^{th\,2}}{2\mu_0} 2\pi \left(\frac{1}{2} - 2\frac{1}{4} + \frac{1}{6} \right) \xi^2$$

$$\Rightarrow \varepsilon^{\psi}_{L,1} = \frac{B_c^{th\,2}}{2\mu_0} \cdot \frac{\pi}{3} \xi^2 \qquad (6.60)$$

Here the first factor represents an energy per volume and the second factor coincides with 1/3 of the area of a flux quantum. Expressing the thermodynamic critical field B_c^{th} by the flux quantum ϕ_0 according to (6.39) we can convert (6.60) into the form

$$\varepsilon^{\psi}_{L,1} = \frac{\pi}{12\mu_0} \left(\frac{\phi_0}{2\pi\lambda_L} \right)^2 \qquad (6.61)$$

Although we have been tempted to ignore (6.58) according to the discussion at the beginning of this subsection, we still take it into account within the ansatz (6.56). To this end we note

$$\frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} \Rightarrow \text{grad}\,\psi \stackrel{(6.56)}{=\!=} \frac{\psi_0}{\xi} \qquad (6.62)$$

Thus, inserting (6.56) in (6.58) yields

$$\varepsilon^{\psi}_{L2} = \int dF \frac{\hbar^2}{2m_S} \frac{-\alpha}{\beta} \frac{1}{\xi^2} = \frac{\alpha^2}{2\beta} 2 \left(-\frac{\hbar^2}{2m_S\alpha} \frac{1}{\xi^2} \right) \pi \xi^2$$

$$\stackrel{(5.69),(6.31)}{=\!=} \frac{B_c^{th\,2}}{2\mu_0} 2\pi \xi^2 \stackrel{(6.39)}{=\!=} \frac{\pi}{2\mu_0} \left(\frac{\phi_0}{2\pi\lambda_L} \right)^2 \qquad (6.63)$$

We conclude that (6.63) turns out to be even larger than (6.61). Thus (6.63) can not be neglected.

6.3.8 Calculation of ε^B_L:

The line energy (6.55) contains two contributions. The first one represents the magnetic energy. And taking into account (5.35), (5.36) the second contribution in (6.55) corresponds to the kinetic energy of the superconducting electrons in the London theory. Applying (4.16), (4.32) and (M2), i.e. $\text{rot}\,\vec{B} = \mu_0 \vec{j}_S$ in the stationary case, we obtain from (6.55)

$$\varepsilon^B_L = \int dF \left\{ \frac{\vec{B}^2}{2\mu_0} + \frac{e_S^2 n_S}{2m_S} \left(\frac{m_S}{e_S^2 n_S} \right)^2 \frac{1}{\mu_0^2} (\text{rot}\,\vec{B})^2 \right\} \qquad (6.64)$$

which reduces with the London penetration length (4.47) to

$$\varepsilon^B_L = \frac{1}{2\mu_0} \int dF \left\{ \vec{B}^2 + \lambda_L^2 (\text{rot}\,\vec{B})^2 \right\} \qquad (6.65)$$

In order to further facilitate the evaluation of (6.65) we apply the vector analytic identity (5.54) with $\vec{u} = \text{rot}\,\vec{B}$ and $\vec{v} = \vec{B}$, yielding

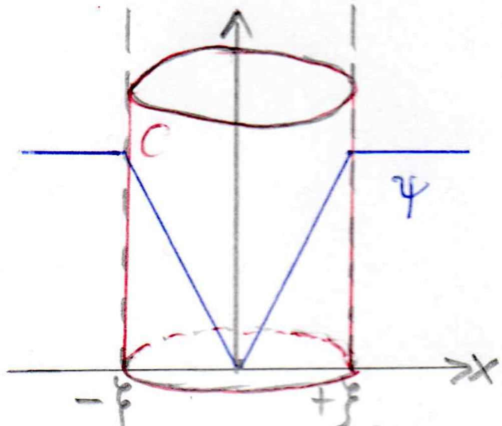
$$\mathcal{E}_L^B = \frac{1}{2\mu_0}\int dF \left\{ \vec{B}^2 + \lambda_L^2\, \vec{B} \cdot \mathrm{rot}(\mathrm{rot}\,\vec{B}) - \lambda_L^2\, \mathrm{div}(\mathrm{rot}\,\vec{B} \times \vec{B}) \right\} \quad (6.65)$$

Applying (6.48) as well as (14.40), (M4) and the Gauß theorem, we decompose (6.65) into a volume and a surface integral:

$$L\,\mathcal{E}_L^B = \frac{1}{2\mu_0}\int_V dV\, \vec{B} \cdot (\vec{B} - \lambda_L^2 \Delta \vec{B}) + \frac{\lambda_L^2}{2\mu_0}\oint_{\partial V} (\vec{B} \times \mathrm{rot}\,\vec{B}) \cdot d\vec{F} \quad (6.66)$$

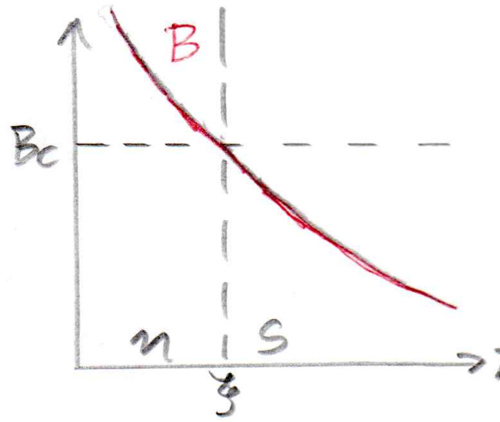
In Section 4.8 we already determined the spatial profile of the magnetic field in the vicinity of the flux quantum in the realm of the London theory. According to (4.124) the magnetic induction is given by the modified Bessel function $K_0(r/\lambda_L)$, which has a logarithmic divergency at the origin. Therefore, the respective integrals in (6.66) turn out to diverge as well.

Thus, in order to obtain a finite result for (6.66) it is necessary from a mathematical point of view to introduce a cut-off cylinder C as sketched in the adjacent figure. Performing the integration over the whole $\mathbb{R}^3$ without the cut-off cylinder, i.e. over $V = \mathbb{R}^3 \setminus C$, the divergency at the origin is avoided.



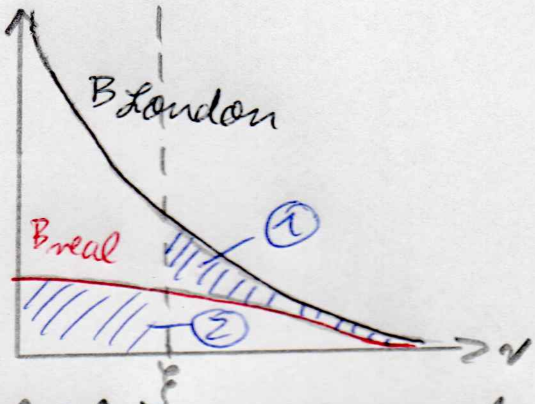
But the choice of this cut-off cylinder C with the cut-off radius ξ in the xy-plane has not only a mathematical background, it can also be justified physically:

1) Argument within London theory:
As the magnetic induction $\vec{B}$ diverges in the limit $r \to 0$, there exists a certain radius $r = \xi$ in the xy-plane, where the critical field B_c is reached. Thus, the superconductor is divided in a normal conducting region for $0 \le r \le \xi$ and a superconducting region for $r \ge \xi$. But as the energy (6.66) is only supposed to be evaluated in the superconducting region, the cut-off cylinder has to be excluded from the integration volume.



2) Argument from taking into account the real profile of the magnetic induction:

Whereas the magnetic induction of the London theory diverges at the origin, the real magnetic induction, following from numerical calculations, turns out to be finite everywhere. Thus, choosing an appropriate cut-off radius $r = \xi$ one may achieve



that the contribution from ①, which is erroneously included in the London theory, corresponds to the contribution from ②, which is neglected. Therefore, one can even expect to obtain a reasonable result from introducing the cut-off cylinder C.

At first, we insert the inhomogeneous Helmholtz equation (4.94) into (6.66):

$$L\,\varepsilon_L^B = \frac{1}{2\mu_0}\int_V dV\, \vec{B}\cdot\vec{e}_z\,\phi_0\,\delta^{(2)}(\vec{x}) + \frac{\lambda_L^2}{2\mu_0}\oint_{\partial V}(\vec{B}\times \mathrm{rot}\,\vec{B})\cdot d\vec{F} \quad (6.67)$$

But as the delta-function has its singularity at the origin, which is excluded from the integration volume $V = \mathbb{R}^3\backslash C$, the volume integral in (6.67) vanishes and only the surface integral remains. Inserting the Oersted law (5.56), (6.67) reduces to

$$L\,\varepsilon_L^B = \frac{\lambda_L^2}{2}\oint_{\partial V}(\vec{B}\times\vec{j}_s)\cdot d\vec{F} \quad (6.68)$$

From (4.124) and (4.125) we read off $\vec{B} = B_z(r)\vec{e}_z$ and $\vec{j}_s = j_\varphi(r)\vec{e}_\varphi$, so taking into account $\vec{e}_z\times\vec{e}_\varphi = -\vec{e}_r$ we obtain

$$L\,\varepsilon_L^B = -\frac{\lambda_L^2}{2}\oint_{\partial V} B_z(r)\,j_\varphi(r)\,\vec{e}_r\cdot d\vec{F} \quad (6.69)$$

Here the area integral is performed along the surface of the cut-off cylinder, which is characterised by $d\vec{F} \sim -\vec{e}_r$:

$$L\,\varepsilon_L^B = \frac{\lambda_L^2}{2}\oint_{\partial V} B_z(r)\,j_\varphi(r)\,dF \quad (6.70)$$

Integrating over the surface of the cylinder, where we have $r=\xi$, the integrand of (6.70) is constant:

$$L\,\varepsilon_L^B = \frac{\lambda_L^2}{2}\,2\pi\xi L\,B_z(\xi)\,j_\varphi(\xi) \Rightarrow \varepsilon_L^B = \pi\xi\lambda_L^2\,B_z(\xi)\,j_\varphi(\xi) \quad (6.71)$$

Due to (4.124) and (4.125) the modified Bessel functions K_0 and K_1 are evaluated at the argument ξ/λ_L, which is small according to the assumption (6.55). Therefore, we approximate (4.124), (4.125) by the behaviour of the modified Bessel functions K_0, K_1 for small arguments, see page 59 at the top:

$$\varepsilon_L^B = \pi \xi \lambda_L^2 \cdot \frac{\phi_0}{2\pi\lambda_L^2}\left(\ln\frac{\lambda_L}{\xi}\right)\frac{\phi_0}{\mu_0 2\pi\lambda_L^3}\cdot\frac{\lambda_L}{\xi} \Rightarrow \varepsilon_L^B = \frac{\pi}{\mu_0}\left(\frac{\phi_0}{2\pi\lambda_L}\right)^2 \ln\frac{\lambda_L}{\xi} \quad (6.72)$$

6.3.9 Lower Critical Field:

Now we add both contributions (6.61), (6.63) for ε_L^ψ and (6.72) for ε_ψ^B in order to obtain the total line energy ε_L and, via (6.52), finally, the lower critical field

$$H_{c1} = \frac{\pi}{\mu_0 \phi_0}\left(\frac{\phi_0}{2\pi\lambda_L}\right)^2\left(\ln\frac{\lambda_L}{\xi} + \frac{7}{12}\right) \quad (6.73)$$

Inserting therein the thermodynamic critical field (6.39), we get

$$B_{c1} = B_c^{th} \cdot \frac{2\sqrt{2}\,\pi\lambda_L\xi}{\phi_0}\,\frac{\pi}{\phi_0}\,\frac{\phi_0^2}{4\pi^2\lambda_L^2}\left(\ln\frac{\lambda_L}{\xi} + \frac{7}{12}\right) = \sqrt{2}\,B_c^{th}\,\frac{\xi}{2\lambda_L}\left(\ln\frac{\lambda_L}{\xi} + \frac{7}{12}\right) \quad (6.74)$$

Thus, introducing the Ginzburg-Landau parameter $\varkappa = \lambda_L/\xi$ we, finally, obtain

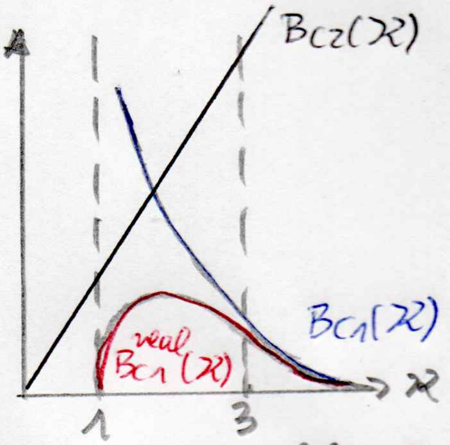
$$B_{c1}(\varkappa) = \sqrt{2}\,B_c^{th}\,\frac{1}{2\varkappa}\left(\ln\varkappa + \frac{7}{12}\right) \quad (6.75)$$

6.3.10 Comparison:

Now we are in the position to discuss how the lower and the upper critical field $B_{c1}(\varkappa)$, $B_{c2}(\varkappa)$ depend on the Ginzburg-Landau parameter $\varkappa$:

$$B_{c1}(\varkappa) \underset{(6.75)}{\sim} \frac{1}{\varkappa}\left(\ln\varkappa + \frac{7}{12}\right)$$

$$B_{c2}(\varkappa) \underset{(6.36)}{\sim} \varkappa$$



If one compares the dependence of the numerically determined $B_{c1}^{real}(\varkappa)$ from $\varkappa$ with (6.75), one recognizes that $B_{c1}(\varkappa)$ from (6.75) is a quite good approximation for $\varkappa \geq 3$.

6.3.11 Example:

In order to provide an illustrative example, we determine the critical fields for the high T_c-superconductor $YBa_2Cu_3O_7$, which is characterized by

$$\lambda_L = 2\cdot 10^{-7}\,m, \quad \varkappa = 100 \quad (6.76)$$

At first we calculate the lower critical field by taking into account the flux quantum (4.89):

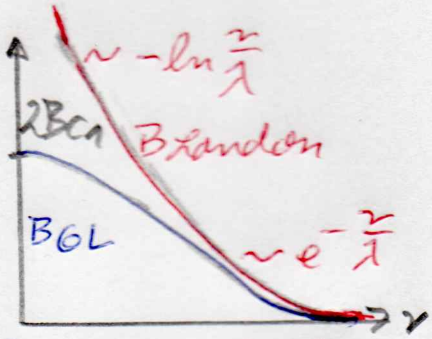
$$B_{c1} \overset{(6.73)}{=} \frac{\Phi_0}{4\pi\lambda_L^2}\left(\ln\kappa + \frac{7}{12}\right) \overset{(4.89),(6.76)}{=} 21\,\mathrm{mT} \qquad (6.77)$$

Then we compare that with the corresponding upper critical field:

$$B_{c2} \overset{(6.36),(6.75)}{=} B_{c1}\,\frac{2\kappa^2}{\ln\kappa + \frac{7}{12}} \overset{(6.76),(6.77)}{=} 81\,\mathrm{T} \qquad (6.78)$$

Thus, for an extreme type II superconductor, B_{c1} is much smaller than B_{c2}. This means that there is a huge range of magnetic induction, in which flux quanta exist in the Shubnikov phase.

6.3.12 Remark:



In the preceding discussion we did not yet clarify how the magnetic induction behaves precisely at the center of the flux quantum. To this end one has to solve the Ginzburg-Landau equations numerically. The result is that the magnetic induction at the core of the flux quantum turns out to be precisely twice the lower critical field:

$$B(0) = 2B_{c1} \qquad (6.79)$$