Quantum Field Theory Problem Sheet 1

Problem 1: Classical Linear Chain

As a preliminary example for classical field theory we consider a system of N point masses M, which are ordered in equilibrium equidistantly within a one-dimensional chain of lattice constant a. Neighboring masses are coupled via elastic springs with a spring constant K. In order to analyze the longitudinal oscillations of this linear chain we introduce the elongations $q_1(t), \dots, q_N(t)$ of the point masses out of their equilibrium positions. As we consider the linear chain as a model for an infinitely extended system, we assume periodic boundary conditions. By demanding

$$q_{N+m}(t) = q_m(t) \tag{1}$$

for the linear chain and any integer m we obtain the topology of a closed ring.

a) Determine the Lagrange function $L(q_1, \dots, q_N; \dot{q}_1, \dots, \dot{q}_N)$ of this system. Derive the underlying equations of evolution for the respective point masses using the Hamilton principle. (2 points)

b) The possible oscillations of such a system are most efficiently analyzed by decomposing the elongations $q_n(t)$ with respect to a suitably chosen set of linear independent basis functions u_n^k:

$$q_n(t) = \sum_k a_k(t)\, u_n^k \,. \tag{2}$$

Here the index k enumerates the set of basis functions. Due to the periodic boundary conditions (1) it is suggested to consider (2) as a discrete Fourier transformation and to choose the basis functions u_n^k via

$$u_n^k = \frac{1}{\sqrt{N}}\, e^{ikna} \,. \tag{3}$$

Show that both (2) and (3) fulfill the period boundary conditions (1) provided the index k is restricted via $k = 2\pi l/(Na)$, where the integer l is given by

$$-\frac{N}{2} < l \leq +\frac{N}{2} \,. \tag{4}$$

Show that the basis functions u_n^k fulfill both the orthonormality relation

$$\sum_{n=1}^{N} u_n^{k*}\, u_n^{k'} = \delta_{kk'} \tag{5}$$

and the completeness relation

$$\sum_k u_n^{k*}\, u_{n'}^{k} = \delta_{nn'} \,. \tag{6}$$

(3 points)

c) Due to (2) and (3) the equations of evolution for the elongations $q_n(t)$ decouple. Show that, consequently, the expansion coefficients $a_k(t)$ in (2) fulfill the differential equation of a harmonic oscillator with frequency ω_k:

$$\ddot{a}_k(t) + \omega_k^2\, a_k(t) = 0 \,. \tag{7}$$

Determine the dispersion relation ω_k. Show that the general solution of (7) for real elongations $q_n(t)$ is given by

$$a_k(t) = b_k\, e^{-i\omega_k t} + b^*_{-k}\, e^{+i\omega_k t}\,, \tag{8}$$

where b_k and b_k^* represent the respective amplitudes. (3 points)

d) Determine the momenta p_n, which are canonical conjugate of the respective elongations q_n, and derive the Hamilton function $H(p_1, \ldots, p_N; q_1, \ldots, q_N)$ of the linear chain. With the help of Eqs. (2), (3), and (8) show that both the elongations $q_n(t)$ and the momenta $p_n(t)$ can be expressed in terms of the amplitudes b_k and b_k^*:

$$\begin{pmatrix} q_n(t) \\ p_n(t) \end{pmatrix} = \sum_k \begin{pmatrix} e^{-i\omega_k t} u_n^k & e^{i\omega_k t} u_n^{k*} \\ -i\omega_k M\, e^{-i\omega_k t} u_n^k & i\omega_k M\, e^{i\omega_k t} u_n^{k*} \end{pmatrix} \begin{pmatrix} b_k \\ b_k^* \end{pmatrix}. \tag{9}$$

Using the orthonormality relation (5) and the dispersion relation ω_k the Hamilton function can be rewritten in terms of the amplitudes b_k and b_k^*. Is the resulting Hamilton function time dependent? (3 points)

Problem 2: Quantum Mechanical Linear Chain

a) Going from the classical to the quantum mechanical treatment of the linear chain the classical observables $q_n(t)$ and $p_n(t)$ become operators in the Heisenberg picture $\hat{q}_n(t)$ and $\hat{p}_n(t)$, for which we have to demand the canonical equal-time commutation relations:

$$[\hat{q}_n(t), \hat{q}_{n'}(t)]_- = [\hat{p}_n(t), \hat{p}_{n'}(t)]_- = 0\,, \qquad [\hat{q}_n(t), \hat{p}_{n'}(t)]_- = i\hbar\, \delta_{nn'}\,. \tag{10}$$

Determine the Hamilton operator $\hat{H}$ of the linear chain. Derive the Heisenberg evolution equations for the operators $\hat{q}_n(t)$ and $\hat{p}_n(t)$.

(2 points)

b) For the quantum mechanical investigation of the linear chain it is useful to also expand the operators $\hat{q}_n(t)$ and $\hat{p}_n(t)$ with respect to the basis functions u_n^k. In analogy with (9) we decompose

$$\begin{pmatrix} \hat{q}_n(t) \\ \hat{p}_n(t) \end{pmatrix} = \sum_k \begin{pmatrix} e^{-i\omega_k t} u_n^k & e^{i\omega_k t} u_n^{k*} \\ -i\omega_k M\, e^{-i\omega_k t} u_n^k & i\omega_k M\, e^{i\omega_k t} u_n^{k*} \end{pmatrix} \begin{pmatrix} \hat{b}_k \\ \hat{b}_k^\dagger \end{pmatrix}, \tag{11}$$

where the classical amplitudes b_k and b_k^* are substituted by their corresponding operators $\hat{b}_k$ and $\hat{b}_k^\dagger$. Explain why this decomposition guarantees that $\hat{q}_n(t)$ and $\hat{p}_n(t)$ are hermitian operators. Use the orthonormality relation in order to reexpress the amplitude operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ conversely in terms of the canonically conjugated operators $\hat{q}_n(t)$ and $\hat{p}_n(t)$. Evaluate the commutator relations

$$\left[\hat{b}_k, \hat{b}_{k'}\right]_- = ?\,, \qquad \left[\hat{b}_k^\dagger, \hat{b}_{k'}^\dagger\right]_- = ?\,, \qquad \left[\hat{b}_k, \hat{b}_{k'}^\dagger\right]_- = ?\,. \tag{12}$$

Rescale the amplitude operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ according to $\hat{b}_k = \alpha_k\, \hat{B}_k$ and $\hat{b}_k^\dagger = \alpha_k\, \hat{B}_k^\dagger$ such that the new operators $\hat{B}_k$ and $\hat{B}_k^\dagger$ fulfill the same equal-time commutator relations as the ladder operators of independent harmonic oscillators. (3 points)

c) By proceeding analogously to Problem 1d) reexpress the Hamilton operator $\hat{H}$ of the quantum mechanical linear chain via the rescaled amplitude operators $\hat{B}_k$ and $\hat{B}_k^\dagger$. Show that in this way you obtain a Hamilton operator for a system of uncoupled harmonic oscillators. (3 points)

d) Define the ground state $|0\rangle$ of the linear chain. What is its expectation value for the energy? (1 point)

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until May 3 at 14.00.