TU KAISERSLAUTERN — SS 2022

Department of Physics — Priv.-Doz. Dr. Axel Pelster

Quantum Field Theory — **Problem Sheet 10**

Problem 19: Identities for Dirac Matrices

In view of evaluating Feynman diagrams it is necessary to investigate useful identities of Dirac matrices. To this end it is not necessary to use the explicit form of the Dirac matrices γ^μ in a particular representation. All identities follow directly from the defining anticommutators of the Clifford algebra $[\gamma^\mu, \gamma^\nu]_+ = 2\, g^{\mu\nu}\, I$, where I denotes the unity matrix.

a) Determine the following contractions of Dirac matrices:

$$\begin{aligned} \gamma_\mu\, \gamma^\mu &= 4\,, && (1)\\ \gamma_\mu\, \not{a}\, \gamma^\mu &= -2\, \not{a}\,, && (2)\\ \gamma_\mu\, \not{a}\, \not{b}\, \gamma^\mu &= 4\, a \cdot b\,, && (3)\\ \gamma_\mu\, \not{a}\, \not{b}\, \not{c}\, \gamma^\mu &= -2\, \not{c}\, \not{b}\, \not{a}\,. && (4) \end{aligned}$$

(4 points)

b) For the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ prove the following properties

$$\begin{aligned} \gamma^5\gamma^5 &= I\,, && (5)\\ \left[\gamma^5\,,\, \gamma^\mu\right]_+ &= 0\,. && (6) \end{aligned}$$

(2 points)

c) With the help of the matrix γ^5 show

$$\mathrm{Tr}\,\left(\not{a}_1\, \not{a}_2 \ldots \not{a}_{2n+1}\right) = 0\,. \qquad (7)$$

Which result do you get then for the trace of products with an odd number of Dirac matrices? (2 points)

d) Prove the following trace identities of the Dirac matrices:

$$\begin{aligned} \mathrm{Tr}\,(\not{a}\, \not{b}) &= 4\, a \cdot b\,, && (8)\\ \mathrm{Tr}\,(\not{a}\, \not{b}\, \not{c}\, \not{d}) &= 4\Big[(a\cdot b)\,(c\cdot d) - (a\cdot c)\,(b\cdot d) + (a\cdot d)\,(b\cdot c)\Big]\,, && (9)\\ \mathrm{Tr}\,\left(\gamma^5\right) &= 0\,, && (10)\\ \mathrm{Tr}\,\left(\gamma^5\, \not{a}\, \not{b}\right) &= 0\,. && (11) \end{aligned}$$

(4 points)

e) Prove that the commutator of Dirac matrices

$$S^{\mu\nu} = \frac{i}{4}\,[\gamma^\mu\,,\, \gamma^\nu]_- \qquad (12)$$

fulfills the commutation relations of the Lorentz algebra:

$$\left[S^{\alpha\beta}\,,\,S^{\gamma\delta}\right]_- = i\left(g^{\alpha\delta}\,S^{\beta\gamma} + g^{\beta\gamma}\,S^{\alpha\delta} - g^{\alpha\gamma}\,S^{\beta\delta} - g^{\beta\delta}\,S^{\alpha\gamma}\right)\,. \tag{13}$$

(3 points)

f) Prove the following contractions

$$\gamma_\lambda\,S^{\mu\nu}\,\gamma^\lambda = 0\,, \tag{14}$$

$$\gamma_\lambda\,S^{\mu\nu}\,\gamma^\kappa\,\gamma^\lambda = 2\gamma^\kappa\,S^{\mu\nu}\,. \tag{15}$$

(2 points)

Problem 20: Discrete Symmetries for the Quantized Dirac Field

In the Dirac theory the field operator $\hat{\psi}(x)$ is decomposed into plane waves with the momentum $\mathbf{p}$ and the spin s according to

$$\hat{\psi}(x) = \sum_{s=\pm1/2}\int d^3p\,\sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\mathbf{p}}}}\left\{e^{-ipx/\hbar}u(\mathbf{p},s)\hat{a}_{\mathbf{p},s} + e^{ipx/\hbar}v(\mathbf{p},s)\hat{b}^\dagger_{\mathbf{p},s}\right\}\,, \tag{16}$$

where we have introduced $p = (E_{\mathbf{p}}/c,\mathbf{p})$ with the relativistic dispersion $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2c^2 + m^2c^4}$. The respective four-spinors $u(\mathbf{p},s)$ und $v(\mathbf{p},s)$ are defined in the Weyl representation as

$$u(\mathbf{p},s) = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\dfrac{p\sigma}{mc}}\chi(s)\\ \sqrt{\dfrac{p\tilde{\sigma}}{mc}}\chi(s)\end{pmatrix}\,,\qquad v(\mathbf{p},s) = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\dfrac{p\sigma}{mc}}\chi^c(s)\\ -\sqrt{\dfrac{p\tilde{\sigma}}{mc}}\chi^c(s)\end{pmatrix} \tag{17}$$

and the two-spinors $\chi(s)$, $\chi^c(s)$ read

$$\chi\left(+\frac{1}{2}\right) = \begin{pmatrix}1\\0\end{pmatrix}\,,\quad \chi\left(-\frac{1}{2}\right) = \begin{pmatrix}0\\1\end{pmatrix}\,, \tag{18}$$

$$\chi^c\left(+\frac{1}{2}\right) = \begin{pmatrix}0\\1\end{pmatrix}\,,\quad \chi^c\left(-\frac{1}{2}\right) = \begin{pmatrix}-1\\0\end{pmatrix}\,. \tag{19}$$

a) Prove the relation

$$u(\tilde{\mathbf{p}},s) = \gamma^0u(\mathbf{p},s)\,,\qquad v(\tilde{\mathbf{p}},s) = -\gamma^0v(\mathbf{p},s)\,, \tag{20}$$

by using (17) and the explicit representation

$$\gamma^0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\,. \tag{21}$$

(1 point)

b) A space inversion is implemented on the level of the creation and annihilation operators by the linear operator $\mathcal{P}$:

$$\mathcal{P}\hat{a}_{\mathbf{p},s}\mathcal{P}^{-1} = \eta_P \hat{a}_{\tilde{\mathbf{p}},s}\,, \qquad \mathcal{P}\hat{a}^\dagger_{\mathbf{p},s}\mathcal{P}^{-1} = \eta_P \hat{a}^\dagger_{\tilde{\mathbf{p}},s}\,, \tag{22}$$

$$\mathcal{P}\hat{b}_{\mathbf{p},s}\mathcal{P}^{-1} = -\eta_P \hat{b}_{\tilde{\mathbf{p}},s}\,, \qquad \mathcal{P}\hat{b}^\dagger_{\mathbf{p},s}\mathcal{P}^{-1} = -\eta_P \hat{b}^\dagger_{\tilde{\mathbf{p}},s}\,, \tag{23}$$

where $\tilde{\mathbf{p}} = -\mathbf{p}$. With the help of (16) and (20) determine the transformed field operator

$$\hat{\psi}'_P(x) = \mathcal{P}\hat{\psi}(x)\mathcal{P}^{-1}\,. \tag{24}$$

Prove that $\hat{\psi}'_P(x)$ will obey the Dirac equation if $\hat{\psi}(x)$ satisfies

$$\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}I\right)\hat{\psi}(x) = 0\,. \tag{25}$$

(2 points)

c) For the 2x2 matrices

$$c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & c \end{pmatrix}\,, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{26}$$

prove the relation

$$c\,\sigma^\mu = (\tilde{\sigma}^\mu)^*\,c\,. \tag{27}$$

Then show

$$v(\mathbf{p},s) = C\bar{u}^T(\mathbf{p},s)\,, \qquad u(\mathbf{p},s) = C\bar{v}^T(\mathbf{p},s)\,, \tag{28}$$

by applying (17)–(19) and the explicit representation (21) and

$$C = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}\,. \tag{29}$$

(3 points)

d) The charge conjugation is implemented on the level of creation and annihilation operators by the linear operator $\mathcal{C}$:

$$\mathcal{C}\hat{a}_{\mathbf{p},s}\mathcal{C}^{-1} = \eta_C \hat{b}_{\mathbf{p},s}\,, \qquad \mathcal{C}\hat{a}^\dagger_{\mathbf{p},s}\mathcal{C}^{-1} = \eta_C \hat{b}^\dagger_{\mathbf{p},s}\,, \tag{30}$$

$$\mathcal{C}\hat{b}_{\mathbf{p},s}\mathcal{C}^{-1} = \eta_C \hat{a}_{\mathbf{p},s}\,, \qquad \mathcal{C}\hat{b}^\dagger_{\mathbf{p},s}\mathcal{C}^{-1} = \eta_C \hat{a}^\dagger_{\mathbf{p},s}\,. \tag{31}$$

With the help of (16) and (28) calculate the transformed field operator

$$\hat{\psi}'_C(x) = \mathcal{C}\hat{\psi}(x)\mathcal{C}^{-1}\,. \tag{32}$$

Prove with (26), (27) and (29) that the matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \tag{33}$$

obey the identities

$$\gamma^\mu = \gamma^0 (\gamma^\mu)^\dagger \gamma^0 , \quad (\gamma^\mu)^T = -C^{-1} \gamma^\mu C . \tag{34}$$

Show then that $\hat{\psi}'_C(x)$ fulfills the Dirac equation (25) provided that $\hat{\psi}(x)$ obeys it. (4 points)

e) For the matrix

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{35}$$

and (29) prove the relations

$$\gamma^5 C u(\mathbf{p}, s) = -(-1)^{1/2-s} u(\tilde{\mathbf{p}}, -s)^* , \; \gamma^5 C v(\mathbf{p}, s) = -(-1)^{1/2-s} v(\tilde{\mathbf{p}}, -s)^* . \tag{36}$$

(2 points)

f) The time inversion is implemented on the level of creation and annihilation operators by the **antilinear** operator $\mathcal{T}$:

$$\mathcal{T} \hat{a}_{\mathbf{p},s} \mathcal{T}^{-1} = \eta_T (-1)^{1/2-s} \hat{a}_{\tilde{\mathbf{p}},-s} , \quad \mathcal{T} \hat{a}^\dagger_{\mathbf{p},\tilde{s}} \mathcal{T}^{-1} = \eta_T (-1)^{1/2-s} \hat{a}^\dagger_{\tilde{\mathbf{p}},-s} , \tag{37}$$

$$\mathcal{T} \hat{b}_{\mathbf{p},s} \mathcal{T}^{-1} = \eta_T (-1)^{1/2-s} \hat{b}_{\tilde{\mathbf{p}},-s} , \quad \mathcal{T} \hat{b}^\dagger_{\mathbf{p},\tilde{s}} \mathcal{T}^{-1} = (-1)^{1/2-s} \eta_T \hat{b}^\dagger_{\tilde{\mathbf{p}},-s} . \tag{38}$$

With the help of (16) and (36) calculate the transformed field operator

$$\hat{\psi}'_T(x) = \mathcal{T} \hat{\psi}(x) \mathcal{T}^{-1} \tag{39}$$

by taking into account the **antilinear property** of $\mathcal{T}$:

$$\mathcal{T} \left(\alpha \hat{a}_{\mathbf{p},s} + \beta \hat{b}^\dagger_{\mathbf{p},s} \right) = \alpha^* \mathcal{T} \hat{a}_{\mathbf{p},s} + \beta^* \mathcal{T} \hat{b}^\dagger_{\mathbf{p},s} . \tag{40}$$

Prove the identity

$$\tilde{\gamma}^\mu = (\gamma^5 C)^{-1} (\gamma^\mu)^* (\gamma^5 C) . \tag{41}$$

Which equation fulfills $\hat{\psi}'_T(x)$ provided that $\hat{\psi}(x)$ solves the Dirac equation (25)?

(3 points)

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until July 5 at 14.00.