TU KAISERSLAUTERN SS 2022
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Quantum Field Theory **Problem Sheet 11**

Problem 21: Electron-Proton Scattering

Consider a scattering process where an electron and a proton in the initial state

$$|i\rangle = |\mathbf{p}_i, s_i; \mathbf{P}_i, S_i\rangle \tag{1}$$

go over into an electron and a proton in the final state

$$|f\rangle = |\mathbf{p}_f, s_f; \mathbf{P}_f, S_f\rangle \,. \tag{2}$$

The corresponding scattering operator $\hat{S}$ is given with the time ordering operator $\hat{T}$ via

$$\hat{S} = \hat{T}\left\{e^{-i\hat{H}_I/\hbar}\right\} , \tag{3}$$

where Hamiltonian operator in the interaction picture

$$\hat{H}_I = \frac{e}{c}\int d^4x \hat{A}_\mu(x)\Big\{\hat{j}_e^\mu(x) - \hat{j}_p^\mu(x)\Big\} \tag{4}$$

contains the normal ordered current density operators

$$\hat{j}_e^\mu = c : \hat{\bar{\psi}}_e(x)\gamma^\mu\hat{\psi}_e(x) : , \qquad \hat{j}_p^\mu = c : \hat{\bar{\psi}}_p(x)\gamma^\mu\hat{\psi}_p(x) : \,. \tag{5}$$

a) Show that the scattering matrix $S_{fi} = \langle f|\hat{S}|i\rangle$ in lowest order is given by

$$S_{fi} = \frac{e^2}{\hbar^2 c^2}\int d^4x \int d^4y \, \langle 0|\hat{T}\Big\{\hat{A}_\mu(x)\hat{A}_\nu(y)\Big\}|0\rangle\langle \mathbf{p}_f, s_f|\hat{j}_e^\mu(x)|\mathbf{p}_i, s_i\rangle \, \langle \mathbf{P}_f, S_f|\hat{j}_p^\nu(y)|\mathbf{P}_i, S_i\rangle \,. \tag{6}$$

(2 points)

b) Insert the Fourier decomposition of both the photon propagator

$$\langle 0|\hat{T}\Big\{\hat{A}_\mu(x)\hat{A}_\nu(y)\Big\}|0\rangle = \frac{i\hbar}{c\epsilon_0}\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \, \frac{-g_{\mu\nu}}{k^2 + i\epsilon} \tag{7}$$

and the field operators

$$\hat{\psi}_e(x) = \sum_{\mathbf{p},s} \sqrt{\frac{mc^2}{Ve_{\mathbf{p}}}} \left\{e^{-ip\cdot x/\hbar} u(\mathbf{p}, s)\hat{a}_{\mathbf{p},s} + e^{ip\cdot x/\hbar} v(\mathbf{p}, s)\hat{b}^\dagger_{\mathbf{p},s}\right\} , \tag{8}$$

$$\hat{\psi}_p(x) = \sum_{\mathbf{P},S} \sqrt{\frac{Mc^2}{VE_{\mathbf{P}}}} \left\{e^{-iP\cdot x/\hbar} u(\mathbf{P}, S)\hat{a}_{\mathbf{P},S} + e^{iP\cdot x/\hbar} v(\mathbf{P}, S)\hat{b}^\dagger_{\mathbf{P},S}\right\} \tag{9}$$

in (6) with the respective relativistic dispersion relations $e_{\mathbf{p}} = \sqrt{\mathbf{p}^2c^2 + m^2c^4}$ and $E_{\mathbf{P}} = \sqrt{\mathbf{P}^2c^2 + M^2c^4}$. Show that the scattering matrix results in

$$\begin{aligned} S_{fi} &= \frac{e^2\hbar}{\epsilon_0 c}(2\pi\hbar)^4\delta(p_f + P_f - p_i - P_i)\sqrt{\frac{mc^2}{Ve_{\mathbf{p}_i}}}\sqrt{\frac{mc^2}{Ve_{\mathbf{p}_f}}}\sqrt{\frac{Mc^2}{VE_{\mathbf{P}_i}}}\sqrt{\frac{Mc^2}{VE_{\mathbf{P}_f}}} \\ &\quad \times \bar{u}(\mathbf{p}_f, s_f)\gamma^\mu u(\mathbf{p}_i, s_i)\frac{-ig_{\mu\nu}}{(p_f - p_i)^2 + i\epsilon}\bar{u}(\mathbf{P}_f, S_f)\gamma^\nu u(\mathbf{P}_i, S_i)\,. \end{aligned} \tag{10}$$

Interpret (10) via Feynman diagrams. (3 points)

c) Justify why the cross section of this scattering process is defined according to

$$\sigma = \frac{1}{4}\sum_{s_i, s_f, S_i, S_f}\int d^3p_f\,\frac{V}{(2\pi\hbar)^3}\int d^3P_f\,\frac{V}{(2\pi\hbar)^3}\,\frac{|S_{fi}|^2}{JT}\,. \tag{11}$$

Insert (10) into (11) and apply the heuristic rule (see lecture)

$$\delta(p_f + P_f - p_i - P_i)^2 = \frac{VTc}{(2\pi\hbar)^4}\delta(p_f + P_f - p_i - P_i)\,. \tag{12}$$

Show that the cross section can then be rewritten as

$$\sigma = \int d^3p_f\int d^3P_f\,\frac{e^4(mc^2)^2(Mc^2)^2M_{fi}}{16\pi^2\epsilon_0^2 cVJe_{\mathbf{p}_i}e_{\mathbf{p}_f}E_{\mathbf{P}_i}E_{\mathbf{P}_f}(p_f - p_i)^4}\delta(p_f + P_f - p_i - P_i) \tag{13}$$

and determine the expression for the spinor contribution M_{fi}. (1 point)

d) The expresion for M_{fi} contains terms of the form ,,adjoint spinor · matrix · spinor". As it represents a complex number, its complex conjugate coincides with its adjoint. Considering the hermiticity of γ^0 and the anti-hermiticity of γ^i prove the identity:

$$\left\{\bar{u}(\mathbf{p}_f, s_f)\gamma^\mu u(\mathbf{p}_i, s_i)\right\}^* = \bar{u}(\mathbf{p}_i, s_i)\gamma^\mu u(\mathbf{p}_f, s_f)\,. \tag{14}$$

(1 point)

e) For both four-spinors

$$u(\mathbf{p}_i, s_i) = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{p_i\sigma}{mc}}\\ \sqrt{\frac{p_i\tilde{\sigma}}{mc}}\end{pmatrix}\chi(s_i)\,, \quad \chi\left(+\frac{1}{2}\right) = \begin{pmatrix}1\\0\end{pmatrix}, \quad \chi\left(-\frac{1}{2}\right) = \begin{pmatrix}0\\1\end{pmatrix} \tag{15}$$

prove the property

$$\sum_{s_i = \pm 1/2} u(\mathbf{p}_i, s_i)\bar{u}(\mathbf{p}_i, s_i) = \frac{\not{p}_i + mc}{2mc}\,. \tag{16}$$

Show that all summations with respect to the spin variables s_i, s_f, S_i, S_f in M_{fi} can be explicitly performed with the help of (16), so that M_{fi} is of the form

$$M_{fi} = \frac{1}{16(mc)^2(Mc)^2}\,\mathrm{Tr}\left\{(\not{p}_f + mc)\gamma^\mu(\not{p}_i + mc)\gamma^\nu\right\}\cdot\mathrm{Tr}\left\{(\not{P}_f + Mc)\gamma_\mu(\not{P}_i + Mc)\gamma_\nu\right\}. \tag{17}$$

(3 points)

f) Determine the traces in (17) with the methods of Problem **19** and show

$$M_{fi} = \frac{2\left[p_f \cdot P_f \, p_i \cdot P_i + p_f \cdot P_i \, p_i \cdot P_f - (mc)^2 P_f \cdot P_i - (Mc)^2 p_f \cdot p_i + 2(mc)^2(Mc)^2\right]}{(mc)^2(Mc)^2} \,. \tag{18}$$

(2 points)

g) Go now into a reference system where the incoming proton rests:

$$E_{\mathbf{P}_i} = Mc^2 \,, \qquad \mathbf{P}_i = \mathbf{0} \,. \tag{19}$$

In the reference system the number of incoming particles per time and area is given by $J = |\vec{j}_e|$ with $j_e^k = \langle i|\hat{j}_e^k|i\rangle$. With the help of (1), (5), (8), (15) and (20) show that the current density then reads

$$J = \frac{|\mathbf{p}_i| c^2}{V e_{\mathbf{p}_i}} \,. \tag{20}$$

Note: Use the identities from the lecture

$$\sqrt{\frac{p_i \sigma}{mc}} \, \tilde{\sigma}^\mu \sqrt{\frac{p_i \sigma}{mc}} = \Lambda^\mu{}_\nu(\mathbf{p}_i) \, \tilde{\sigma}^\nu \,, \tag{21}$$

$$\sqrt{\frac{p_i \tilde{\sigma}}{mc}} \, \sigma^\mu \sqrt{\frac{p_i \tilde{\sigma}}{mc}} = \Lambda^\mu{}_\nu(\mathbf{p}_i) \, \sigma^\nu \,. \tag{22}$$

(2 points)

h) Determine the cross section from (11), (18) – (20) by nelecting the proton recoil. This approximation follows from performing the formal limit $M \to \infty$ so that you have, for instance: $E_{\mathbf{P}_f} \approx Mc^2$. With this approximation show that you obtain the Mott cross section

$$\left(\frac{d\sigma}{d\Omega_f}\right)_{\text{Mott}} = \left(\frac{d\sigma}{d\Omega_f}\right)_{\text{Ruth}} \frac{1 - \beta^2 \sin^2 \theta/2}{1 - \beta^2} \,, \tag{23}$$

$$\left(\frac{d\sigma}{d\Omega_f}\right)_{\text{Ruth}} = \frac{\alpha^2 m^2 \hbar^2 c^2}{4|\mathbf{p}_i|^4 \sin^4 \theta/2} \,. \tag{24}$$

Here the following notations were introduced: Ω_f stands for the solid angle, $\alpha = e^2/(4\pi\epsilon_0\hbar c)$ represents the Sommerfeld fine structure constant, $\beta = |\mathbf{p}_i| c / e_{\mathbf{p}_i}$ denotes the dimensionless velocity, and θ abbreviates the angle between $\mathbf{p}_i$ and $\mathbf{p}_f$. (4 points)

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until July 19 at 14.00.