Quantum Field Theory Problem Sheet 12

Problem 22: Anomalous Magnetic Moment

Consider a process where an electron in the initial state $|\mathbf{p}, \sigma\rangle$ is scattered to the final state $|\mathbf{p}', \sigma'\rangle$. General symmetry considerations show that for such a process the expectation value of the current density is given by the bilinear expression

$$\langle \mathbf{p}', \sigma' | \hat{J}^\mu | \mathbf{p}, \sigma \rangle = ec\, e^{i(p'-p)x/\hbar} \frac{mc^2}{V\sqrt{E_\mathbf{p} E_{\mathbf{p}'}}} \bar{u}(\mathbf{p}', \sigma') \Gamma^\mu(p', p) u(\mathbf{p}, \sigma) \,. \tag{1}$$

Here the vertex function $\Gamma^\mu(p', p)$ has the structure

$$\Gamma^\mu(p', p) = \gamma^\mu F(q^2) + \frac{1}{2mc}(p + p')^\mu G(q^2) \,, \tag{2}$$

where $q = p' - p$ denotes the momentum transfer. The form factors $F(q^2)$, $G(q^2)$ are real, obey the sum rule

$$F(0) + G(0) = 1 \,, \tag{3}$$

and determine the Landé factor of the electron according to

$$g = 2\Big[1 - G(0)\Big] \,. \tag{4}$$

a) In lowest order the electron does not interact with the vacuum and the vertex function is given by

$$\Gamma^{(0)\mu}(p', p) = \gamma^\mu \,. \tag{5}$$

Visualize (5) via a Feynman diagram and identify the corresponding form factors $F^{(0)}(q^2)$, $G^{(0)}(q^2)$. In this order determine the Landé factor. (1 point)

b) The first correction of the vertex function turns out to be given by the following integral:

$$\Gamma^{(1)\mu}(p', p) = \int \frac{d^4k}{(2\pi)^4} \frac{-ie\gamma^\nu}{\hbar} \frac{i\hbar}{\not{p}' - \hbar\not{k} - mc} \gamma^\mu \frac{i\hbar}{\not{p} - \hbar\not{k} - mc} \frac{-ie\gamma^\lambda}{\hbar} \frac{-i\hbar g_{\nu\lambda}}{c\epsilon_0 k^2} \,. \tag{6}$$

Draw the corresponding Feynman diagram. Show that the vertex correction (6) can be rewritten in the form

$$\Gamma^{(1)\mu}(p', p) = \frac{-ie^2\hbar}{c\epsilon_0} \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\nu(\not{p}' - \hbar\not{k} + mc)\gamma^\mu(\not{p} - \hbar\not{k} + mc)\gamma_\nu}{\left[(p' - \hbar k)^2 - (mc)^2\right]\left[(p - \hbar k)^2 - (mc)^2\right] k^2} \,. \tag{7}$$

(1 point)

c) Apply the Clifford algebra

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu} \tag{8}$$

in order to determine contracted products of Dirac matrices

$$\begin{aligned} \gamma^\nu \gamma_\nu &= ?\,, & (9)\\ \gamma^\nu \gamma^\mu \gamma_\nu &= ?\,, & (10)\\ \gamma^\nu \gamma^\lambda \gamma^\mu \gamma_\nu &= ?\,, & (11)\\ \gamma^\nu \gamma^\kappa \gamma^\lambda \gamma^\mu \gamma_\nu &= ?\,. & (12) \end{aligned}$$

With the help of (9)–(12) show that the numerator of the integrand in (7)

$$N^\mu(k) = \gamma^\nu(\not{p}' - \hbar\not{k} + mc)\gamma^\mu(\not{p} - \hbar\not{k} + mc)\gamma_\nu \tag{13}$$

can be rewritten as

$$N^\mu(k) = \gamma^\mu\left[2\hbar^2 k^2 + 4p\cdot p' - 4\hbar k\cdot(p+p')\right] + 4\hbar\not{k}\left[(p+p')^\mu - \hbar k^\mu\right] - 4mc\,\hbar k^\mu\,. \tag{14}$$

To this end use the strategy to move $\not{p}$ to the right and $\not{p}'$ to the left. Due to (1) and the property of the four-spinors

$$(\not{p} - mc)u(\mathbf{p},\sigma) = 0\,, \qquad \bar{u}(\mathbf{p},\sigma)(\not{p} - mc) = 0 \tag{15}$$

you can then substitute $\not{p}$ and $\not{p}'$ by the electron mass m times the light velocity c.

(4 points)

d) Prove the Feynman identity

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^x dy \frac{1}{[Ay + B(x-y) + C(1-x)]^3}\,. \tag{16}$$

Apply (16) to the fraction of the integrand in (7)

$$B(k) = \frac{1}{[(p' - \hbar k)^2 - (mc)^2]\,[(p - \hbar k)^2 - (mc)^2]\,(\hbar k)^2} \tag{17}$$

and by taking into account the mass shell condition

$$p^2 = p'^2 = (mc)^2 \tag{18}$$

show the following result:

$$B(k) = 2\int_0^1 dx \int_0^x dy \frac{1}{\left\{[\hbar k - p'y - p(x-y)]^2 - (mc)^2 x^2 + q^2 y(x-y)\right\}^3}\,. \tag{19}$$

(2 points)

e) Insert (13) and (19) in (7) and perform the substitution

$$\tilde{k}(k) = k - yp'/\hbar - (x-y)p/\hbar \,. \tag{20}$$

Determine with this the integrand $f(\tilde{k}, x, y)$ in the integral representation of the vertex correction:

$$\Gamma^{(1)\mu}(p', p) = \int d^4\tilde{k} \int\limits_0^1 dx \int\limits_0^x dy f(\tilde{k}, x, y) \,. \tag{21}$$

(1 point)

f) Prove the identity

$$\int\limits_0^x dy f(\tilde{k}, x, y) = \int\limits_0^x d\tilde{y} f(\tilde{k}, x, x - \tilde{y}) \,. \tag{22}$$

Combine (21) and (22) in order to rewrite the integral representation of the vertex correction as

$$\Gamma^{(1)\mu}(p', p) = \int d^4\tilde{k} \int\limits_0^1 dx \int\limits_0^x dy \tilde{f}(\tilde{k}, x, y) \,, \tag{23}$$

where the modified integrand is given by

$$\tilde{f}(\tilde{k}, x, y) = \frac{1}{2} \left\{ f(\tilde{k}, x, y) + f(\tilde{k}, x, x - y) \right\} \,. \tag{24}$$

(1 point)

g) The vertex correction (23) turns out to have the structure (2). Thus, you can identify the form factor $G^{(1)}(q^2)$. Show that this implies for $G^{(1)}(0)$ the integral representation

$$G^{(1)}(0) = \frac{8i(mc)^2 e^2}{(2\pi)^4 \hbar c \epsilon_0} \int\limits_0^1 dx \int d^4k \frac{x^2(x-1)}{[k^2 - (mc)^2 x^2]^3} \,. \tag{25}$$

(1 point)

h) The momentum integral in (25) follows from performing a Wick rotation $k^0 = ik^4$, from introducing the Euclidean scalar product $k_{\mathrm{E}}^2 = \mathbf{k}^2 + (k^4)^2$, and from taking into account $d^4k_{\mathrm{E}} = 2\pi^2 k_{\mathrm{E}}^3 dk_{\mathrm{E}}$. With this show the Schwinger result

$$G^{(1)}(0) = -\frac{\alpha}{2\pi} \tag{26}$$

with the Sommerfeld fine structure constant $\alpha = e^2/(4\pi\hbar c\epsilon_0)$ What is then the Landé factor of the electron in this order? Compare this Schwinger result numerically with the current experimental value. (3 points)

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until July 26 at 14.00.