TU KAISERSLAUTERN SS 2022

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Quantum Field Theory **Problem Sheet 3**

Problem 7: Continuum Limit of Classical Linear Chain

Consider similar to Problem Sheet 1 a chain of N point masses with mass M, which are ordered in equilibrium equidistantly within a one-dimensional chain of lattice constant a. Thus, the chain has the length $L = Na$. Two neighboring point masses are coupled by a spring with the spring constant K. But perform now the continuum limit in the limiting process $a \to 0$, where the density $\rho = M/a$ remains constant. Moreover, it turns out to be natural to require in the limiting process $a \to 0$ that $Ka = g$ remains constant.

a) According to Problem Sheet 1 the equation of motion in the discrete case reads

$$M\ddot{q}_n(t) = K\Big[q_{n+1}(t) - 2q_n(t) + q_{n-1}(t) \Big] . \tag{1}$$

Derive the continuum limit for this equation of motion.
Hint: Introduce the continuous elongation $q(x,t)$ by identifying $q_n(t) = q(x = na, t)$ and perform a Taylor expansion of the right-hand side with respect to the lattice constant a in the limiting process $a \to 0$. (2 points)

b) Determine in a similar way the continuum limit of the underlying Lagrangian.
Hint: Use $M = \rho a$ as well as $K = g/a$ and interpret a as dx. (2 points)

c) Use the Lagrangian from **b)** and calculate the Euler-Lagrange equation in the continuum

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{q}(x,t)} = \frac{\delta L}{\delta q(x,t)} . \tag{2}$$

Compare your result with with **a)**. What do you see?
Hint: Performing the functional derivative $\delta/\delta q(x,t)$ you should consider the spatial coordinate x to be varied and the time t to be fixed. (3 points)

d) How is the field-theoretic canonically conjugated momentum $\pi(x,t)$ defined? Show, that the Hamilton function derived from the Lagrange function in **b)** via a Legendre transformation takes the form

$$H = \int_0^L dx \, \left\{ \frac{\pi^2(x,t)}{2\rho} + \frac{g}{2}\left[\frac{\partial q(x,t)}{\partial x}\right]^2 \right\} . \tag{3}$$

(2 points)

e) Determine the Hamilton equations of motion in the continuum limit by exchanging the partial derivatives in the discrete case with corresponding functional derivatives. How do you reobtain the Euler-Lagrange equation from **c)**? (3 points)

Problem 8: Quantum Mechanical Linear Chain: Continuum Limit

Similar to Problem Sheet 1, we treat now the quantum mechanical linear chain in the continuum limit.

a) Introduce the field operators $\hat{q}(x,t)$ and $\hat{\pi}(x,t)$ and quantize with them the Hamiltonian (3). Which commutation relations satisfy the field operators? Determine the Heisenberg equations of motion for the field operators. (4 points)

b) Expand the field operators in discrete Fourier series according to

$$\hat{q}(x,t) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \hat{q}_n(t) e^{ik_n x}, \tag{4}$$

$$\hat{\pi}(x,t) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \hat{\pi}_n(t) e^{ik_n x}. \tag{5}$$

What are suitable values for k_n provided you demand period boundary conditions? Which commutation relations do the Fourier operators $\hat{q}_n(t)$ and $\hat{\pi}_n(t)$ satisfy?
Hint: Use the orthonormality relation

$$\delta_{nm} = \int_0^{2\pi} \frac{dx}{2\pi}\, e^{i(n-m)x} \tag{6}$$

and take into account the hermiteicity $\hat{q}(x,t) = \hat{q}^\dagger(x,t)$ and $\hat{\pi}(x,t) = \hat{\pi}^\dagger(x,t)$. (3 points)

c) Show that in this new basis the Hamiltonian (3) takes the form

$$\hat{H} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\rho}\, \hat{\pi}_n^\dagger \hat{\pi}_n + \frac{\rho}{2}\, \omega_n^2 \hat{q}_n^\dagger \hat{q}_n \right). \tag{7}$$

Which result do you obtain for the frequencies ω_n? What is the physical interpretation of this Hamiltonian? (2 points)

d) Diagonalise the Hamiltonian (7) by introducing creation (annihilation) operators $\hat{a}_n^\dagger$ ($\hat{a}_n$). Check the commutation relations for the creation and annihilation operators explicitly from the ones of the operators $\hat{q}_n$ and $\hat{\pi}_n$. (3 points)

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until May 17 at 14.00.