TU KAISERSLAUTERN SS 2022
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Quantum Field Theory Problem Sheet 4

Problem 9: Schrödinger Field Theory

Here you work out a field-theoretic description of the Schrödinger theory. The underlying action $\mathcal{A}$ is considered to be a functional of both Schrödinger fields $\psi^*(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$. It is defined as a temporal integral over a Lagrange function L according to

$$\mathcal{A} = \int dt\, L\left[\psi^*(\bullet,t), \frac{\partial \psi^*(\bullet,t)}{\partial t}; \psi(\bullet,t), \frac{\partial \psi(\bullet,t)}{\partial t}\right] \tag{1}$$

and the Lagrange function L represents a spatial integral over the Lagrange density $\mathcal{L}$:

$$L = \int d^3x\, \mathcal{L}\left(\psi^*(\mathbf{x},t), \boldsymbol{\nabla}\psi^*(\mathbf{x},t), \frac{\partial \psi^*(\mathbf{x},t)}{\partial t}; \psi(\mathbf{x},t), \boldsymbol{\nabla}\psi(\mathbf{x},t), \frac{\partial \psi(\mathbf{x},t)}{\partial t}\right). \tag{2}$$

Furthermore, the Lagrange density of the Schrödinger field reads

$$\mathcal{L} = i\hbar\, \psi^*(\mathbf{x},t)\, \frac{\partial \psi(\mathbf{x},t)}{\partial t} - \frac{\hbar^2}{2M}\, \boldsymbol{\nabla}\psi^*(\mathbf{x},t) \cdot \boldsymbol{\nabla}\psi(\mathbf{x},t) - V(\mathbf{x})\psi^*(\mathbf{x},t)\psi(\mathbf{x},t)\,. \tag{3}$$

a) Formulate the Hamilton principle of extremizing the action with respect to the Schrödinger fields $\psi^*(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$ and derive with this the corresponding Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \psi^*(\mathbf{x},t)} - \boldsymbol{\nabla}\, \frac{\partial \mathcal{L}}{\boldsymbol{\nabla}\psi^*(\mathbf{x},t)} - \frac{\partial}{\partial t}\, \frac{\partial \mathcal{L}}{\partial \dfrac{\partial \psi^*(\mathbf{x},t)}{\partial t}} = 0\,, \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial \psi(\mathbf{x},t)} - \boldsymbol{\nabla}\, \frac{\partial \mathcal{L}}{\boldsymbol{\nabla}\psi(\mathbf{x},t)} - \frac{\partial}{\partial t}\, \frac{\partial \mathcal{L}}{\partial \dfrac{\partial \psi(\mathbf{x},t)}{\partial t}} = 0\,. \tag{5}$$

(4 points)

b) Evaluate the Euler-Lagrange equations (4), (5) for the Lagrange density (3). What do you get? (2 points)

c) Now we go over from the Lagrange to the Hamilton formulation of classical field theory. Determine to this end at first the momenta fields $\pi^*(\mathbf{x},t)$, $\pi(\mathbf{x},t)$, which are canonically conjugated to the Schödinger fields $\psi^*(\mathbf{x},t)$, $\psi(\mathbf{x},t)$:

$$\pi^*(\mathbf{x},t) = \frac{\delta L}{\delta \dfrac{\partial \psi^*(\mathbf{x},t)}{\partial t}}\,, \qquad \pi(\mathbf{x},t) = \frac{\delta L}{\delta \dfrac{\partial \psi(\mathbf{x},t)}{\partial t}}\,. \tag{6}$$

(2 points)

d) Calculate the Hamilton function follows via a Legendre transformation from the Lagrange function:

$$H = \int d^3x \left\{ \pi^*(\mathbf{x},t)\, \frac{\partial \psi^*(\mathbf{x},t)}{\partial t} + \pi(\mathbf{x},t)\, \frac{\partial \psi(\mathbf{x},t)}{\partial t} \right\} - L\,. \tag{7}$$

Show that it turns out to be of the form

$$H = \int d^3x\, \mathcal{H}\left(\pi(\mathbf{x},t), \boldsymbol{\nabla}\pi(\mathbf{x},t); \psi(\mathbf{x},t), \boldsymbol{\nabla}\psi(\mathbf{x},t)\right)\,. \tag{8}$$

Which result do you get for the Hamilton density $\mathcal{H}$? (3 points)

e) Consider now the action $\mathcal{A}$ as a functional of the fields $\pi(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$, so that the Hamilton principle reads

$$\frac{\delta \mathcal{A}}{\delta \pi(\mathbf{x},t)} = 0\,, \qquad \frac{\delta \mathcal{A}}{\delta \psi(\mathbf{x},t)} = 0\,. \tag{9}$$

Determine from (9) the Hamilton equations of motion of classical field theory and show that they are of the form

$$\frac{\partial \psi(\mathbf{x},t)}{\partial t} = \frac{\partial \mathcal{H}}{\partial \pi(\mathbf{x},t)} - \boldsymbol{\nabla}\, \frac{\partial \mathcal{H}}{\partial \boldsymbol{\nabla}\pi(\mathbf{x},t)}\,, \tag{10}$$

$$\frac{\partial \pi(\mathbf{x},t)}{\partial t} = -\frac{\partial \mathcal{H}}{\partial \psi(\mathbf{x},t)} + \boldsymbol{\nabla}\, \frac{\partial \mathcal{H}}{\partial \boldsymbol{\nabla}\psi(\mathbf{x},t)}\,. \tag{11}$$

Evaluate (10) and (11) for the Hamilton density $\mathcal{H}$ from **d)** and show that you recover with this the equations of motion of the Schrödinger theory. (4 points)

f) The Poisson brackets of two functionals $F\left[\pi(\bullet,\bullet);\psi(\bullet,\bullet)\right]$ and $G\left[\pi(\bullet,\bullet);\psi(\bullet,\bullet)\right]$ is defined via

$$\Big\{F,G\Big\}_- = \int d^3x \left(\frac{\delta F}{\delta \psi(\mathbf{x},t)}\, \frac{\delta G}{\delta \pi(\mathbf{x},t)} - \frac{\delta F}{\delta \pi(\mathbf{x},t)}\, \frac{\delta G}{\delta \psi(\mathbf{x},t)} \right)\,. \tag{12}$$

Evaluate now the Poisson brackets with the Hamilton function

$$\Big\{\psi(\mathbf{x},t), H\Big\}_- = ?\,, \qquad \Big\{\pi(\mathbf{x},t), H\Big\}_- = ? \tag{13}$$

and interpret your result physically. Furthermore, determine the fundamental Poisson brackets of the Schrödinger field $\psi(\mathbf{x},t)$ and its canonical momentum field $\pi(\mathbf{x},t)$ at equal times:

$$\Big\{\psi(\mathbf{x},t), \psi(\mathbf{x}',t)\Big\}_- = ?\,, \qquad \Big\{\pi(\mathbf{x},t), \pi(\mathbf{x}',t)\Big\}_- = ?\,, \qquad \Big\{\psi(\mathbf{x},t), \pi(\mathbf{x}',t)\Big\}_- = ?\,. \tag{14}$$

(4 points)

Problem 10: Canonical Field Quantization of Schrödinger Theory

You work out here the canonical field quantization in the Heisenberg picture. To this end associate to the complex Schrödinger field $\psi(\mathbf{x}, t)$ and its canonically conjugated momentum field $\pi(\mathbf{x}, t)$ corresponding second quantized field operators $\hat{\psi}(\mathbf{x}, t)$ and $\hat{\pi}(\mathbf{x}, t)$. Furthermore, in close analogy to the quantum mechanics for a finite number of degrees of freedom, postulate that the Poisson bracket between two functionals F and G goes over into a commutator between their corresponding second quantized operators $\hat{F}$ and $\hat{G}$ as follows:

$$\left\{F, G\right\}_- \qquad \Longrightarrow \qquad \frac{1}{i\hbar}\left[\hat{F}, \hat{G}\right]_- . \tag{15}$$

a) Starting from the fundamental Poisson brackets (14) determine the resulting equal-time commutation relations

$$\left[\hat{\psi}(\mathbf{x}, t), \hat{\psi}(\mathbf{x}', t)\right]_- = ?\,, \qquad \left[\hat{\pi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)\right]_- = ?\,, \qquad \left[\hat{\psi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)\right]_- = ?\,. \tag{16}$$

Show that you could deduce from **9c)** that the momentum field operator $\hat{\pi}(\mathbf{x}, t)$ is given by the adjoint field operator $\hat{\psi}^\dagger(\mathbf{x}, t)$ via

$$\hat{\pi}(\mathbf{x}, t) = i\hbar\, \hat{\psi}^\dagger(\mathbf{x}, t)\,. \tag{17}$$

Which consequences does (16) have for the equal-time commutation relations between the field operators $\hat{\psi}(\mathbf{x}, t)$ and $\hat{\psi}^\dagger(\mathbf{x}, t)$? (3 points)

b) Deduce from **9e)** and the postulate (15) the Heisenberg equations

$$i\hbar \frac{\partial \hat{\psi}(\mathbf{x}, t)}{\partial t} = \left[\hat{\psi}(\mathbf{x}, t), \hat{H}\right]_- , \tag{18}$$

$$i\hbar \frac{\partial \hat{\pi}(\mathbf{x}, t)}{\partial t} = \left[\hat{\pi}(\mathbf{x}, t), \hat{H}\right]_- . \tag{19}$$

Which explicit form does the Hamilton operator $\hat{H}$ have? Which result do you then get for evaluating the Heisenberg equations? (4 points)

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until May 24 at 14.00.