TU KAISERSLAUTERN SS 2022
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Quantum Field Theory — Problem Sheet 6

Problem 13: Klein-Gordon Equation with Electromagnetic Field

Within a non-relativistic theory the electromagnetic field is described by a scalar potential $\Phi(\mathbf{x}, t)$ and a vector potential $\mathbf{A}(\mathbf{x}, t)$. Coupling a non-relativistic quantum particle with charge e minimally to an electromagnetic field, the usual Jordan rule is modified in SI units according to

$$E \Longrightarrow i\hbar\frac{\partial}{\partial t} - e\Phi(\mathbf{x}, t), \qquad \mathbf{p} \Longrightarrow \frac{\hbar}{i}\boldsymbol{\nabla} - e\mathbf{A}(\mathbf{x}, t)\,. \tag{1}$$

Contrary to that the electromagnetic field is described within the realm of a relativistic theory by a four-potential, whose contravariant components $A^\mu(x^\mu)$ consist of the scalar potential $\Phi(\mathbf{x}, t)$ and the vector potential $\mathbf{A}(\mathbf{x}, t)$:

$$(A^\mu(x^\mu)) = \begin{pmatrix} \Phi(\mathbf{x}, t)/c \\ \mathbf{A}(\mathbf{x}, t) \end{pmatrix}\,. \tag{2}$$

The minimal coupling (1) of a relativistic quantum particle to the electromagnetic field reads then in contravariant notation:

$$p^\mu \Longrightarrow i\hbar\partial^\mu - eA^\mu(x^\mu). \tag{3}$$

a) Start with the relativistic energy-momentum relation $p^\mu p_\mu - (mc)^2 = 0$ and apply the modified Jordan rule (3). Derive with this the Klein-Gordon equation with electromagnetic field for a wave function $\Psi(x^\mu)$. Write it in a compact form by using the gauge covariant derivative

$$D^\mu = \partial^\mu + \frac{ie}{\hbar}A^\mu(x^\mu)\,. \tag{4}$$

(2 points)

b) Electrodynamics is a gauge invariant theory as the Maxwell equations for the electric and the magnetic field do not change with respect to a local gauge transformation of the four-potential

$$A'^\mu(x^\mu) = A^\mu(x^\mu) + \partial^\mu\chi(x^\mu) \tag{5}$$

with an arbitrary gauge function $\chi(x^\mu)$. As the Klein-Gordon wave function $\Psi(x^\mu)$ is uniquely determined only up to a phase factor, the local gauge transformation (5) can be complemented by

$$\Psi'(x^\mu) = \Psi(x^\mu)\,\exp\left\{-\frac{ie}{\hbar}\chi(x^\mu)\right\}\,. \tag{6}$$

How does then the gauge covariant derivative D^μ transform? Prove that the Klein-Gordon equation with electromagnetic field is invariant with respect to the local gauge transformations (5) and (6). (4 points)

c) Derive for the Klein-Gordon equation with electromagnetic field the continuity equation

$$\partial^\mu j_\mu(x^\mu) = 0\,. \tag{7}$$

Determine the covariant components $j_\mu(x^\mu)$ of the four-current density. (3 points)

d) Decomponse the four-potential and the four-spacetime in their respectie time- and space-like parts. Show with this that the Klein-Gordon equation with electromagnetic field gets the following form:

$$\begin{aligned}&\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + \frac{2ie}{\hbar c}\left[\frac{\Phi(\mathbf{x},t)}{c}\frac{\partial}{\partial t} + \mathbf{A}(\mathbf{x},t)\boldsymbol{\nabla}\right] + \frac{ie}{\hbar c}\left[\frac{1}{c}\frac{\partial\Phi(\mathbf{x},t)}{\partial t} + \mathrm{div}\mathbf{A}(\mathbf{x},t)\right]\right.\\ &\left.-\frac{e^2}{\hbar^2}\left[\frac{\Phi(\mathbf{x},t)^2}{c^2} - \mathbf{A}(\mathbf{x},t)^2\right] + \frac{m^2c^2}{\hbar^2}\right\}\Psi(\mathbf{x},t) = 0\,.\end{aligned} \tag{8}$$

(3 points)

Problem 14: Klein Paradox

A relativistic particle of mass m and charge e comes from $x = -\infty$ and hits at $x = 0$ a potential step $V(x,t) = V = e\Phi = \text{constant}$:

a) Specialize the Klein-Gordon equation (8) to this one-dimensional problem. Decompose to this end the wave function $\Psi(x,t)$ for both regions $x < 0$ and $x > 0$:

$$\Psi(x,t) = \begin{cases} \Psi_{\mathrm{I}}(x,t) & ; x < 0, \\ \Psi_{\mathrm{II}}(x,t) & ; x > 0\,. \end{cases} \tag{9}$$

Determine the equations of motion for both $\Psi_{\mathrm{I}}(x,t)$ and $\Psi_{\mathrm{II}}(x,t)$. (1 point)

b) In both equations of motion it is possible to separate the spatial and the temporal part of the wave function:

$$\Psi_{\mathrm{I}}(x,t) = f_{\mathrm{I}}(t)\varphi_{\mathrm{I}}(x) \quad ; x < 0\,, \tag{10}$$

$$\Psi_{\mathrm{II}}(x,t) = f_{\mathrm{II}}(t)\varphi_{\mathrm{II}}(x) \quad ; x > 0\,. \tag{11}$$

Due to this separation ansatz the partial differential equations for $\Psi_{\mathrm{I}}(x,t)$ and $\Psi_{\mathrm{II}}(x,t)$ decompose into ordinary differential equations for the functions $f_I(t), f_{\mathrm{II}}(t), \varphi_{\mathrm{I}}(x), \varphi_{\mathrm{II}}(x)$.

(1 point)

c) Determine the general solutions of the respective ordinary differential equations. In order to determine then the physical solution one has to demand that both the wave function $\Psi(x,t)$ and its first partial derivatives with respect to x and t at $x = 0$ are continuous. Determine with this the functions $f_{\rm I}(t)$ and $f_{\rm II}(t)$. Which intermediate result do you get for the wave function $\Psi(x,t)$? (2 points)

d) Determine now the functions $\varphi_I(x)$ and $\varphi_{\rm II}(x)$, by solving the corresponding ordinary differential equations and by imposing the continuity conditions from **14c)**. Take also into account the boundary condition of the problem that in region $x > 0$ no left-moving wave should exist. Discuss the resulting solution for the following three cases:

$$\text{i) } 0 \leq V \leq E - mc^2, \qquad \text{ii) } E - mc^2 \leq V \leq E + mc^2, \qquad \text{iii) } E + mc^2 \leq V\,. \tag{12}$$

Solve the same problem within the realm of non-relativistic quantum mechanics. Compare the non-relativistic solution with the corresponding relativistic one. (4 points)

e) With the help of the $\mu = 0$-component of the covariant four-current density $j_\mu(x,t)$ from problem **13c)** you can determine via $\rho(x,t) = j_0(x,t)/c$ the charge density $\rho(x,t)$. Determine the charge density $\rho(x,t)$ for both regions $x < 0$ and $x > 0$ and discuss the three cases i) to iii). (2 points)

f) Determine for all three cases i) to iii) in both regions $x < 0$ und $x > 0$ the current $j(x,t)$, which you can identify with the negative of the $\mu = 1$-component of the covariant four-current density $j_\mu(x,t)$ from problem **13c)**. Interpret your results by introducing the transmission coefficient T and the reflection coefficient R. Which result do you get in case iii) if you demand that the group velocity

$$v_g = \frac{1}{\hbar}\frac{dE}{dk} \tag{13}$$

should be positive in the region $x > 0$? Is this result compatible with a one-particle interpretation? (2 points)

g) Solve the Klein paradox by combining the Heisenberg uncertainty relation of quantum mechanics with the relativistic energy-momentum relation. (1 point)

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until June 7 at 14.00.