Quantum Field Theory **Problem Sheet 7**

Problem 15: Quantization of Maxwell Field

The operators of the transversal vector potential $\hat{A}_{\perp i}(\mathbf{x},t)$ and their canonical conjugated momenta $\hat{\pi}_i(\mathbf{x},t) = \epsilon_0 \frac{\partial \hat{A}_{\perp i}(\mathbf{x},t)}{\partial t}$ in the Heisenberg picture obey the equal-time commutation relations

$$\left[\hat{\pi}_i(\mathbf{x},t), \hat{\pi}_j(\mathbf{x}',t)\right]_- = \left[\hat{A}_{\perp i}(\mathbf{x},t), \hat{A}_{\perp j}(\mathbf{x}',t)\right]_- = 0, \qquad \left[\hat{A}_{\perp i}(\mathbf{x},t), \hat{\pi}_j(\mathbf{x}',t)\right]_- = i\hbar\, \delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}') \tag{1}$$

with the transversal delta function

$$\delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}') = \delta_{ij}\delta(\mathbf{x}-\mathbf{x}') + \frac{1}{4\pi}\partial'_j\partial'_i \frac{1}{|\mathbf{x}-\mathbf{x}'|}\,. \tag{2}$$

a) Demonstrate that the commutation relations (1) are compatible with the quantized version of the transversality condition

$$\partial_i \hat{A}_{\perp i}(\mathbf{x},t) = 0\,. \tag{3}$$

Here we have used the Einstein summation convention that implies a summation over equal indices. (2 points)

b) In the lecture it is shown that the field operator $\hat{\mathbf{A}}_{\perp}(\mathbf{x},t)$ has the following Fourier decomposition:

$$\hat{\mathbf{A}}_{\perp}(\mathbf{x},t) = \sum_{\lambda=\pm} \int d^3k N_{\mathbf{k}} \left[\boldsymbol{\epsilon}(\mathbf{k},\lambda) e^{i(\mathbf{k}\mathbf{x}-\omega_{\mathbf{k}}t)} \hat{a}_{\mathbf{k},\lambda} + \boldsymbol{\epsilon}(\mathbf{k},\lambda)^* e^{-i(\mathbf{k}\mathbf{x}-\omega_{\mathbf{k}}t)} \hat{a}^{\dagger}_{\mathbf{k},\lambda}\right] \tag{4}$$

Here the dispersion reads $\omega_{\mathbf{k}} = c|\mathbf{k}|$ and the polarization vectors $\boldsymbol{\epsilon}(\mathbf{k},\lambda)$ fulfill the properties

$$\mathbf{k}\,\boldsymbol{\epsilon}(\mathbf{k},\lambda) = 0\,, \qquad \boldsymbol{\epsilon}(\mathbf{k},\lambda)\,\boldsymbol{\epsilon}(\mathbf{k},\lambda')^* = \delta_{\lambda,\lambda'}\,, \qquad \boldsymbol{\epsilon}(-\mathbf{k},\lambda) = \boldsymbol{\epsilon}(\mathbf{k},-\lambda) = \boldsymbol{\epsilon}(\mathbf{k},\lambda)^*\,. \tag{5}$$

Prove

$$\hat{a}_{\mathbf{k},\lambda} = \frac{1}{2(2\pi)^3 N_{\mathbf{k}}} \int d^3x\, \boldsymbol{\epsilon}(-\mathbf{k},\lambda)\, e^{-i(\mathbf{k}\mathbf{x}-\omega_{\mathbf{k}}t)} \left[\hat{\mathbf{A}}_{\perp}(\mathbf{x},t) + \frac{i}{\epsilon_0\omega_{\mathbf{k}}}\hat{\boldsymbol{\pi}}(\mathbf{x},t)\right] \tag{6}$$

and the corresponding relation for $\hat{a}^{\dagger}_{\mathbf{k},\lambda}$. (4 points)

c) Determine the commutators

$$\left[\hat{a}_{\mathbf{k},\lambda}\,,\, \hat{a}_{\mathbf{k}',\lambda'}\right]_- =?\,, \qquad \left[\hat{a}^{\dagger}_{\mathbf{k},\lambda}\,,\, \hat{a}^{\dagger}_{\mathbf{k}',\lambda'}\right]_- =?\,, \qquad \left[\hat{a}_{\mathbf{k},\lambda}\,,\, \hat{a}^{\dagger}_{\mathbf{k}',\lambda'}\right]_- =?\,. \tag{7}$$

How do you have to fix the normalization constants $N_{\mathbf{k}}$, so that the operators $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$ can be interpreted as annihilation and creation operators of a single particle? (6 points)

d) According to the lecture the Hamilton operator of the electromagnetic field reads

$$\hat{H} = \frac{1}{2}\int d^3x \left[\frac{1}{\epsilon_0}\,\hat{\pi}_i(\mathbf{x},t)\hat{\pi}_i(\mathbf{x},t) + \frac{1}{\mu_0}\,\partial_i\hat{A}_{\perp j}(\mathbf{x},t)\partial_i\hat{A}_{\perp j}(\mathbf{x},t)\right]. \tag{8}$$

Show that the normal ordered Hamilton operator $:\hat{H}: = \hat{H} - \langle 0|\hat{H}|0\rangle$ has the following representation:

$$:\hat{H}: = \sum_{\lambda=\pm 1}\int d^3k\,\hbar\omega_{\mathbf{k}}\,\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}\,. \tag{9}$$

(4 points)

e) The momentum operator of the electromagnetic field reads

$$\hat{\mathbf{p}} = \int d^3x\,\left[\boldsymbol{\nabla}\times\hat{\mathbf{A}}_\perp(\mathbf{x},t)\right]\times\hat{\boldsymbol{\pi}}(\mathbf{x},t)\,. \tag{10}$$

Demonstrate that the normal ordered momentum operator $:\hat{\mathbf{p}}: = \hat{\mathbf{p}} - \langle 0|\hat{\mathbf{p}}|0\rangle$ has the following representation

$$:\hat{\mathbf{p}}: = \sum_{\lambda=\pm 1}\int d^3k\,\hbar\mathbf{k}\,\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}\,. \tag{11}$$

(4 points)

f) The spin operator of the electromagnetic field reads

$$\hat{\mathbf{S}} = \int d^3x\,\hat{\mathbf{A}}_\perp(\mathbf{x},t)\times\hat{\boldsymbol{\pi}}(\mathbf{x},t)\,. \tag{12}$$

Prove that the normal ordered spin operator $:\hat{\mathbf{S}}: = \hat{\mathbf{S}} - \langle 0|\hat{\mathbf{S}}|0\rangle$ possesses the following representation

$$:\hat{\mathbf{S}}: = \sum_{\lambda=\pm 1}\int d^3k\,\lambda\hbar\,\frac{\mathbf{k}}{|\mathbf{k}|}\,\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}\,. \tag{13}$$

Hint: Use the following identity for the polarization vectors

$$\boldsymbol{\epsilon}(\mathbf{k},\lambda)\times\boldsymbol{\epsilon}(\mathbf{k},\lambda')^* = -i\lambda\,\frac{\mathbf{k}}{|\mathbf{k}|}\,\delta_{\lambda\lambda'}\,. \tag{14}$$

(4 points)

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until June 14 at 14.00.