TU KAISERSLAUTERN SS 2022

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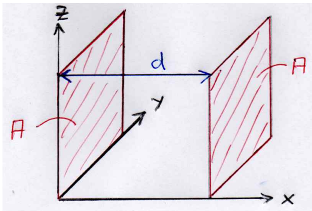
Quantum Field Theory **Problem Sheet 8**

Problem 16: Casimir Effect

Consider two metal plates of area $A = L_y L_z$, which have the distance d:



a) Write down the four Maxwell equations in vacuum. Show by applying at the interface of vacuum and metal plates both the Gauß and the Stokes law that the following boundary conditions have to be fulfilled:

$$\begin{aligned} \mathbf{e}_x \cdot \mathbf{B} &= 0\,, \qquad \text{for } x = 0 \text{ and } x = d\,, \\ \mathbf{e}_x \times \mathbf{E} &= \mathbf{0}\,, \qquad \text{for } x = 0 \text{ and } x = d\,. \end{aligned}$$

(3 points)

b) Show that, solving this boundary value problem, the Maxwell theory admits two types of standing-wave solutions, which read with the transversal wave vector $\mathbf{k}_\perp = k_y \mathbf{e}_y + k_z \mathbf{e}_z$:

- Transversal electric (TE) modes, i.e. $\mathbf{e}_x \cdot \mathbf{E} = 0$:

$$\begin{aligned} \mathbf{E}(x,y,z,t) &= N_{\rm TE}\, i\omega\, \sin(k_x x)\, \mathbf{e}_x \times \mathbf{k}_\perp\, e^{i(k_y y + k_z z - \omega t)}\,, \\ \mathbf{B}(x,y,z,t) &= N_{\rm TE}\, \left[i k_\perp^2 \sin(k_x x) \mathbf{e}_x - k_x \cos(k_x x) \mathbf{k}_\perp \right] e^{i(k_y y + k_z z - \omega t)}\,. \end{aligned}$$

- Transversal magnetic (TM) modes, i.e. $\mathbf{e}_x \cdot \mathbf{B} = 0$:

$$\begin{aligned} \mathbf{B}(x,y,z,t) &= N_{\rm TM}\, \frac{\omega}{c^2}\, \cos(k_x x)\, \mathbf{e}_x \times \mathbf{k}_\perp\, e^{i(k_y y + k_z z - \omega t)}\,, \\ \mathbf{E}(x,y,z,t) &= N_{\rm TM}\, \left[i k_x \sin(k_x x) \mathbf{k}_\perp - k_\perp^2 \cos(k_x x) \mathbf{e}_x \right] e^{i(k_y y + k_z z - \omega t)}\,. \end{aligned}$$

Which discrete values do you get in both cases for the x-component k_x of the wave vector $\mathbf{k}$?

(6 points)

c) The thermodynamic limit of infinitely large lengths L_y, L_z yields a nearly continuous set of states denoted by the transversal wave vectors $\mathbf{k}_\perp$ with the transverse density of states $A/(2\pi)^2$, where the area is given by $A = L_y L_z$. Determine that with this you obtain for the vacuum energy between the two metal plates

$$E_{\text{plates}}^{\text{inside}} = 2\sum_{n=0}^{\infty}\left(1-\frac{1}{2}\,\delta_{n,0}\right) A \int \frac{d^2k_\perp}{(2\pi)^2}\,\frac{1}{2}\,\hbar c\sqrt{\frac{\pi^2 n^2}{d^2}+\mathbf{k}_\perp^2}\,.$$

Show that outside of two metal plates, where the wave vectors are not restricted at all, one yields for a length $L_x \gg d$ the vacuum energy

$$E_{\text{plates}}^{\text{outside}} = 2(L_x - d)A\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\int\frac{d^2k_\perp}{(2\pi)^2}\,\frac{1}{2}\,\hbar c\sqrt{k_x^2+\mathbf{k}_\perp^2}\,.$$

Derive in absence of the metal plates the total vacuum energy in the box with volume $L_x A$:

$$E_{\text{total}} = 2L_x A\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\int\frac{d^2k_\perp}{(2\pi)^2}\,\frac{1}{2}\,\hbar c\sqrt{k_x^2+\mathbf{k}_\perp^2}\,.$$

(3 points)

d) Determine now that the Casimir energy $E_{\text{C}} = E_{\text{plates}}^{\text{inside}} + E_{\text{plates}}^{\text{outside}} - E_{\text{total}}$ is of the form

$$E_{\text{C}} = B\int_0^{\infty} d\tau\,\tau^{-5/2}\left(\sum_{n=0}^{\infty}{}' e^{-\pi^2 n^2\tau/d^2} - \int_0^{\infty} dn\,e^{-\pi^2 n^2/d^2}\right),$$

where we have introduced the abbreviation

$$\sum_{n=0}^{\infty}{}' f_n = \sum_{n=0}^{\infty} f_n - \frac{1}{2}\,f_0\,.$$

What do you get for the constant B. **Hint:** Use the Schwinger trick

$$\frac{1}{a^x} = \frac{1}{\Gamma(x)}\int_0^{\infty} d\tau\,\tau^{x-1}\,e^{-a\tau}\,.$$

(4 points)

e) Prove the distributional identity

$$\sum_{n=-\infty}^{\infty}\delta(x-n) = \sum_{m=-\infty}^{\infty} e^{-2\pi i m x}$$

by determining the Fourier transform of the comb function on the left-hand side. Show with this the Poisson sum formula

$$\left(\sum_{n=0}^{\infty}{}' - \int_0^{\infty} dn\right) f(n) = \sum_{m=1}^{\infty}\,\text{Re}\int_{-\infty}^{\infty} dx\,f(x)\,e^{-2\pi i m x}\,, \qquad f(-x) = f(x)\,.$$

(4 points)

f) Evaluate the Casimir energy (1) with the help of **e)**. **Hint:** Use $\Gamma(-1/2) = -2\sqrt{\pi}$ and $\zeta(4) = \pi^4/90$ with the Riemann zeta function $\zeta(x) = \sum_{n=1}^{\infty} 1/n^x$. (2 points)

g) Is the resulting Casimir force $F_{\text{C}} = -\partial E_{\text{C}}/\partial d$ between the two plates repulsive or attractive? Which Casimir pressure do you get for the distance $d = 1\ \mu$m? Provided that the area is $A = 1(\mu\text{m})^2$, which value does the Casimir force F_{C} have and is it measurable? (2 points)

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until June 21 at 14.00.