TU KAISERSLAUTERN
Department of Physics

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Quantum Field Theory **Problem Sheet 9**

Problem 17: Foldy-Wouthuysen Representation for Field-Free Case

A free spin 1/2 particle with mass m is described in Dirac representation via a four-spinor ψ, which obeys the evolution equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}_{\mathrm{D}} \psi \tag{1}$$

with the Hamilton operator

$$\hat{H}_{\mathrm{D}} = c\vec{\alpha}\hat{\vec{p}} + \beta m c^2 \,. \tag{2}$$

Here α_j with $j = 1, 2, 3$ and β denote the 4×4 matrices

$$\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} , \qquad \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} , \tag{3}$$

which consist of 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{4}$$

as well as the 2×2 identity matrix $\mathbf{1}$ and the 2×2 zero matrix $\mathbf{0}$.

a) Go now over from the Dirac representation to another representation by applying the unitary transformation

$$\phi = \hat{T}\psi \,, \qquad \hat{T} = e^{i\hat{S}} \tag{5}$$

with a hermitian operator $\hat{S}$. To this end transform the equation of motion (1) and determine the transformed Hamilton operator. (1 point)

b) With this you arrive at the Foldy-Wouthuysen representation by choosing

$$\hat{S} = -if(\hat{\vec{p}})\beta\vec{\alpha}\hat{\vec{p}}, \tag{6}$$

where $f(\hat{\vec{p}})$ denotes a yet unknown function of the momentum operator $\hat{\vec{p}}$. Is the operator $\hat{S}$ hermitian? Explicitly determine the unitary operator $\hat{T} = e^{i\hat{S}}$ by expanding the exponential function into a Taylor series. (2 points)

c) State the Hamilton operator $\hat{H}_{\mathrm{FW}}$ in the Foldy-Wouthuysen representation. How do you have to choose the yet unknown function $f(\hat{\vec{p}})$, so that the Hamilton operator $\hat{H}_{\mathrm{FW}}$ becomes diagonal? Which expression do you then get for the diagonalized Hamilton operator $\hat{H}_{\mathrm{FW}}$? (3 points)

d) Determine the orthonormalized four-spinors in the Foldy-Wouthuysen representation, which correspond to plane waves. Afterwards, calculate those plane waves in the original Dirac representation. (3 points)

Problem 18: Foldy-Wouthuysen Representation with Electromagnetic Field

Provided that a spin 1/2 particle with mass m and charge q is coupled to a vector potential $\vec{A}$ and a scalar potential ϕ, the Hamilton operator in the Dirac representation in the SI units system reads

$$\hat{H} = c\vec{\alpha}\left(\vec{p} - q\vec{A}\right) + \beta mc^2 + q\phi I\,, \tag{7}$$

where I is the 4×4 identity matrix. In the non-relativistic limit one can approximately construct a unitary transformation, where the upper and the lower two components of the four-spinor are decoupled from each other. Here, one distinguishes two kinds of operators. Odd (even) operators like α_j (β,I) are those, which (do not) couple to the upper and the lower components.

a) Determine the Taylor series of the operator function

$$\hat{H}'(\lambda) = e^{i\lambda\hat{S}}\left(\hat{H} - i\hbar\frac{\partial}{\partial t}\right)e^{-i\lambda\hat{S}} \tag{8}$$

and with this prove the Baker-Hausdorff identity

$$\hat{H}' = e^{i\hat{S}}\left(\hat{H} - i\hbar\frac{\partial}{\partial t}\right)e^{-i\hat{S}} = \hat{H} + i\left[\hat{S},\hat{H}\right]_- + \frac{i^2}{2}\left[\hat{S},\left[\hat{S},\hat{H}\right]_-\right]_- + \frac{i^3}{6}\left[\hat{S},\left[\hat{S},\left[\hat{S},\hat{H}\right]_-\right]_-\right]_-$$
$$+\frac{i^4}{24}\left[\hat{S},\left[\hat{S},\left[\hat{S},\left[\hat{S},\hat{H}\right]_-\right]_-\right]_-\right]_- + \ldots - \hbar\left(\dot{\hat{S}} + \frac{i}{2}\left[\hat{S},\dot{\hat{S}}\right]_- + \frac{i^2}{6}\left[\hat{S},\left[\hat{S},\dot{\hat{S}}\right]_-\right]_- + \ldots\right). \tag{9}$$

(3 points)

b) Now perform the unitary transformation

$$\hat{T} = e^{i\hat{S}}\,, \qquad \hat{S} = \frac{-i}{mc^2}\beta\hat{O}\,, \tag{10}$$

where $\hat{O}$ denotes a yet unknown odd operator with the anti-commuting property $[\beta,\hat{O}]_+ = 0$. Expand the transformed Hamilton operator (9) in powers of $1/mc^2$ up to the order $(1/mc^2)^3$. How do you have to fix the operator $\hat{O}$, so that $\hat{H}'$ only consists of even operators up to the order $(1/mc^2)^0$? Show that the transformed Hamilton operator $\hat{H}'$ has the form

$$\hat{H}' = \beta mc^2 + q\phi I + \frac{a_1}{mc^2}\beta\hat{U}^2 + \frac{a_2}{m^2c^4}\left[\hat{U},\left[\hat{U},q\phi\right]_-\right]_- + \frac{i\hbar a_3}{m^2c^4}\left[\hat{U},\dot{\hat{U}}\right]_- + \frac{a_4}{m^3c^6}\beta\hat{U}^4 + \frac{a_5}{mc^2}\beta\left[\hat{U},q\phi\right]_-$$
$$+\frac{i\hbar a_6}{mc^2}\beta\dot{\hat{U}} + \frac{a_7}{m^2c^4}\hat{U}^3 + \frac{a_8}{m^3c^6}\beta\left[\hat{U},\left[\hat{U},\left[\hat{U},q\phi\right]_-\right]_-\right]_- + \frac{i\hbar a_9}{m^3c^6}\beta\left[\hat{U},\left[\hat{U},\dot{\hat{U}}\right]_-\right]_- \tag{11}$$

with the odd operator

$$\hat{U} = c\vec{\alpha}\left(\hat{\vec{p}} - q\vec{A}\right) \tag{12}$$

and determine the coefficients $a_1, \ldots, a_9$. (4 points)

c) Perform further successive unitary transformations

$$\hat{T}' = e^{i\hat{S}'}, \qquad \hat{S}' = \frac{-i}{m^2c^4}\beta\hat{O}', \tag{13}$$

$$\hat{T}'' = e^{i\hat{S}''}, \qquad \hat{S}'' = \frac{-i}{m^3c^6}\beta\hat{O}'', \tag{14}$$

$$\hat{T}''' = e^{i\hat{S}'''}, \qquad \hat{S}''' = \frac{-i}{m^4c^8}\beta\hat{O}''', \tag{15}$$

so that the transformed Hamilton operators $\hat{H}''$, $\hat{H}'''$, $\hat{H}''''$ up to the orders $(1/mc^2)^1$, $(1/mc^2)^2$, $(1/mc^2)^3$ consist only of even operators. Determine how the transformed Hamilton operators $\hat{H}''$, $\hat{H}'''$, $\hat{H}''''$ depend on the odd operators $\hat{U}$ up to the order $(1/mc^2)^3$. (4 points)

d) Specialize your result for $\hat{H}''''$ by substituting $\hat{U}$ according to (12) and by introducing the electromagnetic fields $\vec{E} = -\text{ grad } \phi - \dot{\vec{A}}$ and $\vec{B} = \text{ rot } \vec{A}$. How can you physically interpret the respective terms in $\hat{H}''''$? (4 points)

Hint: Show at first $(\vec{\alpha}\vec{A})(\vec{\alpha}\vec{B}) = \vec{A}\vec{B} + i\vec{\Sigma}(\vec{A} \times \vec{B})$, where the vector of the 4×4 spin matrices $\vec{\Sigma}$ is given by

$$\Sigma_j = \begin{pmatrix} \sigma_j & \mathbf{0} \\ \mathbf{0} & \sigma_j \end{pmatrix}. \tag{16}$$

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until June 28 at 14.00.