4 Klein-Gordon Field

Motivation:

"1st" quantization: one classical particle $\longrightarrow$ one quantum particle

$$[\hat{x}_i, \hat{p}_j]_- = i\hbar\, \delta_{ij}$$

"2nd" quantization: one quantum particle $\longrightarrow$ many quantum particles

$$[\hat{\Psi}(\vec{x},t), \hat{\pi}(\vec{x}',t)]_- = i\hbar\, \delta(\vec{x}-\vec{x}')$$

4.4 Canonical Field Quantization:

$$\mathcal{L} = \frac{\hbar^2}{2mc^2}\frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial t} - \frac{\hbar^2}{2m}\vec{\nabla}\Psi^* \cdot \vec{\nabla}\Psi - \frac{1}{2}mc^2\Psi^*\Psi$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi^*}{\partial t}} = \frac{\hbar^2}{2mc^2}\frac{\partial \Psi}{\partial t}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi}{\partial t}} = \frac{\hbar^2}{2mc^2}\frac{\partial \Psi^*}{\partial t}$$

Hamilton density:

$$\mathcal{H} = \pi^*\frac{\partial \Psi^*}{\partial t} + \pi\frac{\partial \Psi}{\partial t} - \mathcal{L} = \frac{2mc^2}{\hbar^2}\underline{\pi^*\pi} + \frac{\hbar^2}{2m}\underline{\vec{\nabla}\Psi^*\cdot\vec{\nabla}\Psi} + \frac{mc^2}{2}\underline{\Psi^*\Psi}$$

Hamilton function: $H = \int d^3x \, \mathcal{H}$

"2nd" quantization:

$\Psi(\vec{x},t), \; \pi(\vec{x},t)$

$\downarrow$

$\hat{\Psi}(\vec{x},t), \; \hat{\pi}(\vec{x},t)$

$$[\hat{\Psi}(\vec{x},t), \hat{\Psi}(\vec{x}',t)]_- = 0 = [\hat{\pi}(\vec{x},t), \hat{\pi}(\vec{x}',t)]_-$$

$$[\hat{\Psi}(\vec{x},t), \hat{\pi}(\vec{x}',t)]_- = i\hbar\,\delta(\vec{x}-\vec{x}')$$

$\Psi^*(\vec{x},t), \; \pi^*(\vec{x},t)$

$\downarrow$

$\hat{\Psi}^+(\vec{x},t), \; \hat{\pi}^+(\vec{x},t)$

$$[\hat{\Psi}^+(\vec{x},t), \hat{\Psi}^+(\vec{x}',t)]_- = 0 = [\hat{\pi}^+(\vec{x},t), \hat{\pi}^+(\vec{x}',t)]_-$$

$$[\hat{\Psi}^+(\vec{x},t), \hat{\pi}^+(\vec{x}',t)]_- = i\hbar\,\delta(\vec{x}-\vec{x}')$$

$$[\hat{\Psi}(\vec{x},t), \hat{\Psi}^+(\vec{x}',t)]_- = 0 = [\hat{\Psi}(\vec{x},t), \hat{\pi}^+(\vec{x}',t)]_-$$

$$[\hat{\pi}(\vec{x},t), \hat{\Psi}^+(\vec{x}',t)]_- = 0 = [\hat{\pi}(\vec{x},t), \hat{\pi}^+(\vec{x}',t)]_-$$

$$\hat{H} = \int d^3x \left\{ \frac{2mc^2}{\hbar^2} \hat{\pi}^+(\vec{x},t)\hat{\pi}(\vec{x},t) + \frac{\hbar^2}{2m} \vec{\nabla}\hat{\Psi}^+(\vec{x},t)\cdot\vec{\nabla}\hat{\Psi}(\vec{x},t) + \frac{mc^2}{2} \hat{\Psi}^+(\vec{x},t)\hat{\Psi}(\vec{x},t) \right\}$$

Note: No operator ordering here!

Heisenberg equations:

$$i\hbar \frac{\partial \hat{\Psi}(\vec{x},t)}{\partial t} = [\hat{\Psi}(\vec{x},t), \hat{H}]_- = \frac{\delta \hat{H}}{\delta \hat{\pi}(\vec{x},t)} = \frac{2mc^2}{\hbar^2} \hat{\pi}^+(\vec{x},t) \qquad (1)$$

$$[\hat{A}, \hat{B}\hat{C}]_- = [\hat{A}, \hat{B}]_- \hat{C} + \hat{B}[\hat{A}, \hat{C}]_-$$

$$i\hbar \frac{\partial \hat{\Phi}^+(\vec{x},t)}{\partial t} = \frac{2mc^2}{\hbar^2} \hat{\pi}(\vec{x},t) \qquad (2)$$

$$i\hbar \frac{\partial \hat{\pi}(\vec{x},t)}{\partial t} = [\hat{\pi}(\vec{x},t), \hat{H}]_- \Rightarrow \frac{\partial \hat{\pi}(\vec{x},t)}{\partial t} = \frac{\hbar^2}{2m} \Delta \hat{\Phi}^+(\vec{x},t) - \frac{mc^2}{2} \hat{\Phi}^+(\vec{x},t) \quad (3)$$

$$i\hbar \frac{\partial \hat{\pi}^+(\vec{x},t)}{\partial t} = [\hat{\pi}^+(\vec{x},t), \hat{H}]_- \Rightarrow \frac{\partial \hat{\pi}^+(\vec{x},t)}{\partial t} = \quad - \quad - \qquad (2)$$

(1) + (4) or (2) + (3):

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right) \hat{\Phi}^+(\vec{x},t) = 0 \qquad (I)$$

Field operator fulfills Klein-Gordon equation.

4.5 Plane Waves:

$$\hat{\Phi}(\vec{x},t) = \int d^3p \underbrace{N_{\vec{p}}}_{\text{normalization constants}} \underbrace{\hat{a}_{\vec{p}}(t)}_{\text{Fourier operators}} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \qquad (II)$$

(II) in (I):
$$\left(\frac{\partial^2}{\partial t^2} + \frac{\vec{p}^2c^2 + m^2c^4}{\hbar^2}\right)\hat{a}_{\vec{p}}(t) = 0$$

special relativistic dispersion: $E_{\vec{p}} = \sqrt{\vec{p}^2c^2 + m^2c^4} = E_{-\vec{p}}$

$$\hat{a}_{\vec{p}}(t) = \hat{a}^{(1)}_{\vec{p}} e^{-\frac{i}{\hbar}E_{\vec{p}}t} + \hat{a}^{(2)}_{\vec{p}} e^{+\frac{i}{\hbar}E_{\vec{p}}t}$$

$$\hat{\Phi}(\vec{x},t) = \int d^3p\, N_{\vec{p}} \left\{ \hat{a}^{(1)}_{\vec{p}} e^{\frac{i}{\hbar}(\vec{p}\vec{x} - E_{\vec{p}}t)} + \underbrace{\hat{a}^{(2)}_{\vec{p}} e^{\frac{i}{\hbar}(\vec{p}\vec{x} + E_{\vec{p}}t)}}_{\vec{p} \to -\vec{p},\ N_{\vec{p}} \xrightarrow{\text{assumption}} N_{-\vec{p}}} \right\}$$

$$= \underbrace{\hat{a}^{(2)}_{-\vec p}}_{=: \hat{a}^{(2)}_{\vec p}} \, e^{-\frac{i}{\hbar}(+\vec p\cdot\vec x + E_{\vec p} t)}$$

$$\Rightarrow \quad \hat\Phi(\vec x,t) = \sum_{r=1}^{2}\int d^3p\, \hat a^{(r)}_{\vec p} \underbrace{N_{\vec p}\, e^{\varepsilon_r \frac{i}{\hbar}(\vec p\vec x - E_{\vec p}t)}}_{= u^{(r)}_{\vec p}(\vec x,t)} \quad ; \quad \varepsilon_r = \begin{cases}+1 ; & r=1\\ -1 ; & r=2\end{cases}$$

$$u^{(r)*}_{\vec p}(\vec x,t) = N_{\vec p}\, e^{(-\varepsilon_r)\frac{i}{\hbar}(\vec p\vec x - E_{\vec p}t)}$$

Fix normalization constant:

$$\langle u^{(r)}_{\vec p}, u^{(r')}_{\vec p'}\rangle = \underline{\varepsilon_r \,\delta_{r,r'}\,\delta(\vec p - \vec p')}$$

$$= \frac{i\hbar}{2mc^2}\int d^3x \left\{ u^{(r)*}_{\vec p}(\vec x,t)\frac{\partial u^{(r')}_{\vec p'}(\vec x,t)}{\partial t} - \frac{\partial u^{(r)*}(\vec x,t)}{\partial t}\, u^{(r')}_{\vec p'}(\vec x,t)\right\}$$

$$= \frac{i\hbar}{2mc^2}\,\frac{-i}{\hbar}\,(\varepsilon_r E_{\vec p} + \varepsilon_{r'} E_{\vec p'})\, N_{\vec p}N_{\vec p'}\, e^{\frac{i}{\hbar}(\varepsilon_r E_{\vec p} - \varepsilon_{r'}E_{\vec p'})t}$$

$$\underbrace{\int d^3x\; e^{\frac{i}{\hbar}(\varepsilon_{r'}\vec p' - \varepsilon_r \vec p)\vec x}}_{=(2\pi\hbar)^3\, \underbrace{\delta(\varepsilon_{r'}\vec p' - \varepsilon_r\vec p)}_{=\delta(\vec p' - \varepsilon_r\varepsilon_{r'}\vec p)}}$$

$1 = \varepsilon_{r'}\varepsilon_{r'}$

$\delta(ax) = \frac{1}{|a|}\delta(x)$

$E_{\vec p} = E_{-\vec p}$

$=$

$N_{\vec p} = N_{-\vec p}$

$$\frac{(2\pi\hbar)^3 E_{\vec p}}{mc^2}\; \underbrace{\frac{\varepsilon_r+\varepsilon_{r'}}{2}}\; N_{\vec p}^2\; \cancel{e^{\frac{i}{\hbar}(\varepsilon_r - \varepsilon_{r'})E_{\vec p}t}}\; \delta(\vec p' - \underbrace{\varepsilon_r\varepsilon_{r'}}_{=+1}\vec p)$$

$$= \begin{cases}\varepsilon_r ; & r=r'\\ 0 ; & r\neq r'\end{cases} \quad = \delta_{r,r'}$$

$$= \boxed{\frac{(2\pi\hbar)^3 E_{\vec{p}}}{Mc^2} N_{\vec{p}}^2} \varepsilon_\nu \delta_{\nu,\nu'} \delta(\vec{p}\,'-\vec{p}) \quad \Rightarrow \quad N_{\vec{p}} = \sqrt{\frac{1}{(2\pi\hbar)^3}\frac{Mc^2}{E_{\vec{p}}}}$$

$$= 1$$

Remark: $u_{\vec{p}}^{(1)*}(\vec{x},t) = u_{\vec{p}}^{(2)}(\vec{x},t)$, $u_{\vec{p}}^{(2)*}(\vec{x},t) = u_{\vec{p}}^{(1)}(\vec{x},t)$

$$\langle u_{\vec{p}}^{(\nu)*}, u_{\vec{p}\,'}^{(\nu')} \rangle = (-\varepsilon_\nu)\, \delta_{\nu,\nu'}\, \delta(\vec{p}-\vec{p})$$

4.6 Fourier Operators:

What is their physical interpretation?

$$\hat{\Psi}(\vec{x},t) = \sum_{\nu=1}^{2} \int d^3p\, \underbrace{\hat{a}_{\vec{p}}^{(\nu)}}_{=?} u_{\vec{p}}^{(\nu)}(\vec{x},t); \quad \hat{\Psi}^+(\vec{x},t) = \sum_{\nu=1}^{2} \int d^3p\, \underbrace{\hat{a}_{\vec{p}}^{(\nu)+}}_{=?} u_{\vec{p}}^{(\nu)*}(\vec{x},t)$$

$$\langle u_{\vec{p}}^{(\nu)}, \hat{\Psi} \rangle = \sum_{\nu'=1}^{2} \int d^3p'\, \hat{a}_{\vec{p}\,'}^{(\nu')} \underbrace{\langle u_{\vec{p}}^{(\nu)}, u_{\vec{p}\,'}^{(\nu')} \rangle}_{= \varepsilon_\nu \delta_{\nu,\nu'} \delta(\vec{p}-\vec{p}\,')} = \varepsilon_\nu \hat{a}_{\vec{p}}^{(\nu)}$$

$$\Rightarrow \boxed{\hat{a}_{\vec{p}}^{(\nu)} = \varepsilon_\nu \langle u_{\vec{p}}^{(\nu)}, \hat{\Psi} \rangle}$$

analogous $\boxed{\hat{a}_{\vec{p}}^{(\nu)+} = -\varepsilon_\nu \langle u_{\vec{p}}^{(\nu)*}, \hat{\Psi}^+ \rangle}$

$$\hat{a}_{\vec{p}}^{(\nu)} = \frac{i\hbar\varepsilon_\nu}{2Mc^2} \int d^3x \left[u_{\vec{p}}^{(\nu)*}(\vec{x},t) \underbrace{\frac{\partial \hat{\Psi}(\vec{x},t)}{\partial t}} - \frac{\partial u_{\vec{p}}^{(\nu)*}(\vec{x},t)}{\partial t} \boxed{\hat{\Psi}(\vec{x},t)} \right]$$

$$= \frac{2mc^2}{\hbar^2} \hat{\pi}^+(\vec{x},t)$$

$$\hat{a}^{(r)+}_{\vec{p}} = \frac{-i\hbar\varepsilon_r}{2mc^2} \int d^3x \left\{ a^{(r)}_{\vec{p}}(\vec{x},t) \frac{2mc^2}{\hbar^2} \hat{\pi}(\vec{x},t) - \frac{\partial a^{(r)}(\vec{x},t)}{\partial t} \hat{\Phi}^+(\vec{x},t) \right\}$$

$$[\hat{a}^{(r)}_{\vec{p}}, \hat{a}^{(r')}_{\vec{p}'}]_- = 0 = [\hat{a}^{(r)+}_{\vec{p}}, \hat{a}^{(r')+}_{\vec{p}'}]_-$$

$$[\hat{a}^{(r)}_{\vec{p}}, \hat{a}^{(r')+}_{\vec{p}'}]_- = \ldots = \boxed{\varepsilon_r} \delta_{r,r'} \delta(\vec{p}-\vec{p}')$$

4.7 Hamilton-Operator:

$$\hat{H} = \int d^3x \left\{ \frac{2mc^2}{\hbar^2} \hat{\pi}^+(\vec{x},t)\hat{\pi}(\vec{x},t) + \frac{\hbar^2}{2m} \vec{\nabla}\Phi^+ \vec{\nabla}\Phi + \frac{mc^2}{2} \Phi^+ \Phi \right\}$$

$$\hat{\Phi}(\vec{x},t) = \sum_{r=1}^{2} \int d^3p \, \hat{a}^{(r)}_{\vec{p}} \underbrace{u^{(r)}_{\vec{p}}(\vec{x},t)}_{= \sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec{p}}}} \, e^{\frac{i}{\hbar}\varepsilon_r(\vec{p}\vec{x} - E_{\vec{p}}t)}}$$

$$\hat{\Phi}^+(\vec{x},t) = \sum_{r=1}^{2} \int d^3p \, \hat{a}^{(r)+}_{\vec{p}} u^{(r)*}_{\vec{p}}(\vec{x},t)$$

$$\hat{\pi}(\vec{x},t) = \frac{\hbar^2}{2mc^2} \frac{\partial \hat{\Phi}^+(\vec{x},t)}{\partial t} = \frac{\hbar^2}{2mc^2} \sum_{r=1}^{2} \int d^3p \, \frac{i}{\hbar} \varepsilon_r E_{\vec{p}} \, \hat{a}^{(r)+}_{\vec{p}} u^{(r)*}_{\vec{p}}(\vec{x},t)$$

$$\hat{\pi}^+ = \ldots$$

$$\hat{H} = \sum_{r=1}^{2} \sum_{r'=1}^{2} \int d^3p \int d^3p' \, \hat{a}^{(r)+}_{\vec{p}} \hat{a}^{(r')}_{\vec{p}'} \underbrace{\int d^3x \, u^{(r)*}_{\vec{p}}(\vec{x},t) \, u^{(r')}_{\vec{p}'}(\vec{x},t)}_{\frac{mc^2}{(2\pi\hbar)^3 \sqrt{E_{\vec{p}} E_{\vec{p}'}}} e^{\frac{i}{\hbar}(\varepsilon_r \cancel{E_{\vec{p}}} - \varepsilon_{r'} E_{\vec{p}'})t} (2\pi\hbar)^3 \delta(\vec{p}' - \varepsilon_r\varepsilon_{r'}\vec{p})}$$

$$= E_{\vec{p}}$$

$$\cdot \left\{ \cancel{\frac{2mc^2}{\hbar^2}} \left(\cancel{\frac{\hbar^2}{2mc^2}}\right)^{\cancel{2}} \cancel{\frac{1}{\hbar^2}} \varepsilon_r E_{\vec{p}} \varepsilon_{r'} E_{\vec{p}'} + \frac{\hbar^2}{2m} \frac{1}{\hbar^2} \underbrace{\varepsilon_r \varepsilon_{r'} \vec{p} \cdot \vec{p}'}_{\vec{p}'^2} + \frac{mc^2}{2} \right\}$$

$$\vec{p} \cdot \vec{p}' \;(\mathrm{lw}) = \varepsilon_r \varepsilon_{r'} \vec{p}$$

$$\frac{\varepsilon_r \varepsilon_{r'} E_{\vec{p}}^2}{2mc^2} + \underbrace{\frac{\vec{p}^2 c^2}{2mc^2} + \frac{mc^2}{2}}_{= \frac{1}{2} E_{\vec{p}}^2} = \frac{\varepsilon_r \varepsilon_r + 1}{2mc^2} E_{\vec{p}}^2$$

$$\frac{\varepsilon_r \varepsilon_{r'} + 1}{2} = \begin{cases} 1 & ; \quad r = r' \\ 0 & \quad r \neq r' \end{cases} = \delta_{r,r'}$$

$$\Rightarrow \hat{H} = \sum_{r=1}^{2} \int d^3p \; E_{\vec{p}} \; \hat{a}^{(r)+}_{\vec{p}} \hat{a}^{(r)}_{\vec{p}}$$

4.8 Charge Operator:

$$Q = \int d^3x \langle \Phi, \Phi \rangle = \frac{i\hbar}{2mc^2} \int d^3x \left\{ \Phi^*(\vec{x},t) \underbrace{\frac{\partial \Phi(\vec{x},t)}{\partial t}}_{\sim \pi^*} - \underbrace{\frac{\partial \Phi^*(\vec{x},t)}{\partial t}}_{\sim \pi} \Phi(\vec{x},t) \right)$$

$\Big\downarrow$ 2nd quantization

$$\hat{Q} = \frac{i}{\hbar} \int d^3x \left\{ \underline{\hat{\Phi}^+(\vec{x},t)} \; \underline{\hat{\pi}^+(\vec{x},t)} - \underline{\hat{\pi}(\vec{x},t)} \; \underline{\hat{\Phi}(\vec{x},t)} \right)$$

Note: Operator ordering matters, demand: $[\hat{Q}, \hat{H}]_- = 0$,

i.e. charge conservation

$$\hat{Q} = \text{-----} = \sum_{r=1}^{2} \int d^3p \, \varepsilon_r \, \hat{a}^{(r)+}_{\vec{p}} \hat{a}^{(r)}_{\vec{p}}$$

4.8 Redefinition of Fourier Operators:

$$[\hat{a}^{(1)}_{\vec{p}}, \hat{a}^{(1)+}_{\vec{p}'}]_- = \delta(\vec{p} - \vec{p}') \qquad [\hat{a}^{(2)}_{\vec{p}}, \hat{a}^{(2)+}_{\vec{p}'}]_- = -\delta(\vec{p} - \vec{p}')$$

annihilator creator creator annihilator

suggest redefinition:

first particle sort: $\hat{a}_{\vec{p}} = \hat{a}^{(1)}_{\vec{p}}$, $\hat{a}^{+}_{\vec{p}} = \hat{a}^{(1)+}_{\vec{p}}$

second " " : $\hat{b}_{\vec{p}} = \hat{a}^{(2)+}_{\vec{p}}$, $\hat{b}^{+}_{\vec{p}} = \hat{a}^{(2)}_{\vec{p}}$

all other commutators are unchanged by this redefinition

$$\hat{\Phi}(\vec{x},t) = \int d^3p \left\{ \hat{a}_{\vec{p}} u_{\vec{p}}(\vec{x},t) + \hat{b}^{+}_{\vec{p}} u^{*}_{\vec{p}}(\vec{x},t) \right\}$$

$$\hat{\Phi}^{+}(\vec{x},t) = \int d^3x \left\{ \hat{a}^{+}_{\vec{p}} u^{*}_{\vec{p}}(\vec{x},t) + \hat{b}_{\vec{p}} u_{\vec{p}}(\vec{x},t) \right\}$$

$$\hat{H} = \int d^3p \, E_{\vec{p}} \left(\hat{a}^{+}_{\vec{p}} \hat{a}_{\vec{p}} + \hat{b}_{\vec{p}} \hat{b}^{+}_{\vec{p}} \right) = \int d^3p \, E_{\vec{p}} \left(\hat{a}^{+}_{\vec{p}} \hat{a}_{\vec{p}} + \hat{b}^{+}_{\vec{p}} \hat{b}_{\vec{p}} \right) + \delta(\vec{0}) \int d^3p \, E_{\vec{p}}$$

$$\hat{Q} = \int d^3p \left(\hat{a}^{+}_{\vec{p}} \hat{a}_{\vec{p}} - \hat{b}_{\vec{p}} \hat{b}^{+}_{\vec{p}} \right) = \int d^3p \left(\hat{a}^{+}_{\vec{p}} \hat{a}_{\vec{p}} - \hat{b}^{+}_{\vec{p}} \hat{b}_{\vec{p}} \right) - \delta(\vec{0}) \int d^3p$$

$$= \underbrace{\hat{b}^{+}_{\vec{p}} \hat{b}_{\vec{p}} + \delta(\vec{0})}$$

"double infinities"

vacuum state: $\hat{b}_{\vec{p}} |0\rangle = \hat{a}_{\vec{p}} |0\rangle = 0$

$$\langle 0 | \hat{H} | 0 \rangle = \delta(\vec{0}) \int d^3p \, E_{\vec{p}}$$

$$\langle 0 | \hat{Q} | 0 \rangle = -\delta(\vec{0}) \int d^3p$$

$$:\hat{H}: = \hat{H} - \langle 0 | \hat{H} | 0 \rangle$$

$$:\hat{Q}: = \hat{Q} - \langle 0 | \hat{Q} | 0 \rangle$$