4.10 Definition of Propagator:

Review:

2nd quantization of Klein-Gordon theory:

particle sort $a \;\hat{=}\; \pi^+$	particle sort $b \;\hat{=}\; \pi^-$
$[\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}'}]_- = 0 = [\hat{a}^+_{\vec{p}}, \hat{a}^+_{\vec{p}'}]_-$	$[\hat{b}_{\vec{p}}, \hat{b}_{\vec{p}'}]_- = 0 = [\hat{b}^+_{\vec{p}}, \hat{b}^+_{\vec{p}'}]_- = 0$
$[\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{p}'}]_- = \delta(\vec{p}-\vec{p}')$	$[\hat{b}_{\vec{p}}, \hat{b}^+_{\vec{p}'}]_- = \delta(\vec{p}-\vec{p}')$

$$[\hat{a}_{\vec{p}}, \hat{b}_{\vec{p}'}]_- = 0 = [\hat{a}_{\vec{p}}, \hat{b}^+_{\vec{p}'}]_- = [\hat{a}^+_{\vec{p}}, \hat{b}_{\vec{p}'}]_- = [\hat{a}^+_{\vec{p}}, \hat{b}^+_{\vec{p}'}]_-$$

$$:\hat{H}: = \hat{H} - \langle 0|\hat{H}|0\rangle = \int d^3p \; \underbrace{E_{\vec{p}}}_{=\sqrt{\vec{p}^2c^2+m^2c^4}} \left(\hat{a}^+_{\vec{p}}\hat{a}_{\vec{p}} + \hat{b}^+_{\vec{p}}\hat{b}_{\vec{p}}\right)$$

normal ordered form

$$:\hat{Q}: = \hat{Q} - \langle 0|\hat{Q}|0\rangle = \int d^3p \left(+\hat{a}^+_{\vec{p}}\hat{a}_{\vec{p}} - \hat{b}^+_{\vec{p}}\hat{b}_{\vec{p}}\right)$$

$$\hat{\Phi}(\vec{x},t) = \int d^3p \left\{\hat{a}_{\vec{p}}\, u_{\vec{p}}(\vec{x},t) + \hat{b}^+_{\vec{p}}\, u^*_{\vec{p}}(\vec{x},t)\right\}$$

$$\hat{\Phi}^+(\vec{x},t) = \int d^3p \left\{\underbrace{\hat{a}^+_{\vec{p}}}_{\text{creation of a particle with }+1}\, u^*_{\vec{p}}(\vec{x},t) + \underbrace{\hat{b}_{\vec{p}}}_{\text{annihilation}}\, \underbrace{u_{\vec{p}}(\vec{x},t)}_{\sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec{p}}}}\; e^{\frac{i}{\hbar}(\vec{p}\vec{x}-E_{\vec{p}}t)}}\right\}$$

# of particle with charge − 1

Introduction:

- Propagators in QFT are needed to describe interacting quantum fields perturbatively. They represent building block of Feynman diagrams.
- Overview:

Schrödinger propagator (non-relativistic many-body theory) $\xleftarrow{\text{non-relativistic limit}}$ Klein-Gordon propagator (scalar QED) $\xrightarrow{\text{derivatives (formal ground)}}$ Dirac propagator (QED)

Definition:

$$G(\vec{x},t;\vec{x}',t') = \langle 0| \boxed{\underbrace{\hat{T}}_{\text{time ordering}}\left(\hat{\Psi}(\vec{x},t)\,\hat{\Psi}^{+}(\vec{x}',t')\right)} |0\rangle \qquad \hat{a}_{\vec{p}}|0\rangle = 0 = \hat{b}_{\vec{p}}|0\rangle$$

$$= \vec{x},t \longleftarrow \vec{x}',t'$$

$\vec{x},t$: annihilate a particle with charge −1

$\vec{x}',t'$: create a particle with charge +1

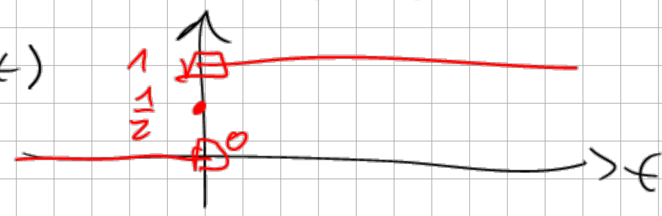
Note: Appears naturally in working out perturbation theory.

$$= \Theta(t-t')\,\hat{\Psi}(\vec{x},t)\,\hat{\Psi}^{+}(\vec{x}',t') \underbrace{+}_{\text{bosonic version}} \Theta(t'-t)\,\hat{\Psi}^{+}(\vec{x}',t')\,\hat{\Psi}(\vec{x},t)$$

$\hat{=}$ operator at later time comes left

$\Theta(t) = \begin{cases} 1 \\ 0 \end{cases}$ Heaviside function

$$\frac{d\Theta(t)}{dt} = \delta(t)$$



Note: Be careful when evaluating a propagator at limit $t \to t'$ as $t \downarrow t'$ and $t \uparrow t'$ may lead to different results.

But this does not happen here at Klein-Gordon theory.

$$G(\vec{x},t;\vec{x}',t') = \Theta(t-t')\,\langle 0|\,\hat{\Phi}(\vec{x},t), \hat{\Phi}^+(\vec{x}',t')\,]\,|0\rangle$$

$$+\Theta(t'-t)\,\langle 0|\,\hat{\Phi}^+(\vec{x}',t')\,\hat{\Phi}(\vec{x},t)\,|0\rangle$$

Question: Equation of motion?

$$\frac{\partial G(\vec{x},t;\vec{x}',t')}{\partial t} = \cancel{\delta(t-t')\,\langle 0|\underbrace{[\hat{\Phi}(\vec{x},t), \hat{\Phi}^+(\vec{x}',t)]_-}_{=0}|0\rangle}$$

$$+\Theta(t-t')\langle 0|\frac{\partial\hat{\Phi}(\vec{x},t)}{\partial t}\hat{\Phi}^+(\vec{x}',t')|0\rangle + \Theta(t'-t)\langle 0|\hat{\Phi}^+(\vec{x}',t')\frac{\partial\hat{\Phi}(\vec{x},t)}{\partial t}|0\rangle$$

$$\frac{\partial^2 G(\vec{x},t;\vec{x}',t')}{\partial t^2} = \delta(t-t')\langle 0|\,[\underbrace{\frac{\partial\hat{\Phi}(\vec{x},t)}{\partial t}}_{=\frac{2mc^2}{\hbar^2}\hat{\pi}^+(\vec{x},t)}, \hat{\Phi}^+(\vec{x}',t)]_-\,|0\rangle$$

$$[\hat{\pi}^+(\vec{x},t), \hat{\Phi}^+(\vec{x}',t)] = -i\hbar\,\delta(\vec{x}-\vec{x}')$$

$$+\Theta(t-t')\langle 0|\underbrace{\frac{\partial^2\hat{\Phi}(\vec{x},t)}{\partial t^2}}\hat{\Phi}^+(\vec{x}',t')|0\rangle + \Theta(t'-t)\langle 0|\hat{\Phi}^+(\vec{x}',t')\underbrace{\frac{\partial^2\hat{\Phi}(\vec{x},t)}{\partial t^2}}|0\rangle$$

$$= \left(c^2 \Delta - \frac{m^2 c^4}{\hbar^2}\right) \hat{\Phi}(\vec{x},t)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right) G(\vec{x},t;\vec{x}',t') = -i\,\frac{2m}{\hbar}\,\delta(t-t')\,\delta(\vec{x}-\vec{x}')$$

Klein-Gordon propagator = Green function of Klein-Gordon equation

<u>Non-relativistic limit:</u>

$$\underset{\uparrow \text{ KG prop.}}{G(\vec{x},t;\vec{x}',t')} = \underset{\uparrow \text{ Schrödinger propagator}}{g(\vec{x},t;\vec{x}',t')} \cdot e^{-\frac{i}{\hbar}mc^2(t-t')}$$

$$\Big(\underbrace{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}_{\downarrow\, c\to\infty \atop 0} - 2\frac{i}{\hbar}m\cancel{c^2}\frac{1}{\cancel{c^2}}\underline{\underline{\frac{\partial}{\partial t}}} \cancel{-\left(\frac{mc^2}{\hbar}\right)^2\frac{1}{c^2}} - \Delta \cancel{+\frac{m^2c^2}{\hbar^2}}\Big) g(\vec{x},t;\vec{x}',t') =$$

$$= \underline{-i\,\frac{2m}{\hbar}}\,\delta(t-t')\,\delta(\vec{x}-\vec{x}')$$

$$\Rightarrow \left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta\right)\underbrace{g(\vec{x},t;\vec{x}',t')} = i\hbar\,\delta(t-t')\,\delta(\vec{x}-\vec{x}')$$

Schrödinger propagator = Green function of Schrödinger equation

4.11 Interpretation of Propagator:

$$\hat{Q} = \frac{i}{\hbar}\int d^3x \left\{ \hat{\Phi}^+(\vec{x},t)\,\hat{\pi}^+(\vec{x},t) - \hat{\pi}(\vec{x},t)\,\hat{\Phi}(\vec{x},t) \right\}$$

$$[\hat{Q}, \hat{\Phi}(\vec{x},t)]_- = \underline{-\hat{\Phi}(\vec{x},t)}\,, \qquad [\hat{Q}, \hat{\Phi}^+(\vec{x},t)]_- = +\hat{\Phi}^+(\vec{x},t)$$

eigenvalue problem: $\hat{Q}\,|q\rangle = q\,|q\rangle$

$$\hat{Q}\left\{\hat{\Phi}(\vec{x},t)\,|q\rangle\right\} = \hat{\Phi}(\vec{x},t)\,(\hat{Q}-1)\,|q\rangle = (q-1)\left\{\hat{\Phi}(\vec{x},t)\,|q\rangle\right\}$$

$$\Rightarrow \quad \hat{\Phi}(\vec{x},t)\,|q\rangle \sim |q-1\rangle$$

charge is reduced by 1

$$\hat{\Phi}^+(\vec{x},t)\,|q\rangle \sim |q+1\rangle$$

charge is increased by 1

$$G(\vec{x},t;\vec{x}',t') = \Theta(t-t')\,\langle 0|\,\hat{\Phi}(\vec{x},t)\,\hat{\Phi}^+(\vec{x}',t')\,|0\rangle$$

propagation of charge +1 from (x',t') to (x,t)

$t > t'$: forward propagation in time

$$+\,\Theta(t'-t)\,\langle 0|\,\hat{\Phi}^+(\vec{x}',t')\,\hat{\Phi}(\vec{x},t)\,|0\rangle$$

propagation of charge −1 from $(\vec{x},t)$ to $(\vec{x}',t')$

$t < t'$: backward propagation in time

4.12 Calculation of Propagator

Use Fourier decomposition of $\hat{\Phi}$, $\hat{\Phi}^+$

$$G(\vec{x},t;\vec{x}',t') = \int d^3p \int d^3p' \,\Big\{\, \Theta(t-t')$$

$$\cdot \underbrace{\langle 0 | \left\{ \hat{a}_{\vec{p}} u_{\vec{p}}(\vec{x},t) + \cancel{\hat{b}^+_{\vec{p}} u^*_{\vec{p}}(\vec{x},t)} \right\} \left\{ \hat{a}^+_{\vec{p}'} u^*_{\vec{p}'}(\vec{x}',t') + \cancel{\hat{b}_{\vec{p}'} u_{\vec{p}'}(\vec{x}',t')} \right\} | 0 \rangle} + \underline{(t \leftrightarrow t')}$$

$$\langle 0 | \underbrace{\hat{a}_{\vec{p}} \hat{a}^+_{\vec{p}'}}_{= \cancel{\langle 0 | \hat{a}^+_{\vec{p}'} \hat{a}_{\vec{p}} | 0 \rangle} + \delta(\vec{p} - \vec{p}')} | 0 \rangle u_{\vec{p}}(\vec{x},t) \, u^*_{\vec{p}'}(\vec{x}',t')$$

$$\underbrace{\langle 0 | 0 \rangle}_{=1}$$

$$= \int d^3p \left\{ \Theta(t-t') \, u_{\vec{p}}(\vec{x},t) u^*_{\vec{p}}(\vec{x}',t') + \Theta(t'-t) \, u_{\vec{p}}(\vec{x}',t') u^*_{\vec{p}}(\vec{x},t) \right\}$$

propagation of a-particles   propagation of b-particles

$$= \int d^3p \sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec{p}}}} \left\{ \Theta(t-t') \, e^{\frac{i}{\hbar}\left[\underline{\vec{p}(\vec{x}-\vec{x}')} - E_{\vec{p}} \underbrace{(t-t')}_{=|t-t'|}\right]} + \Theta(t'-t) \, \underbrace{e^{\frac{i}{\hbar}\left[\underbrace{-\vec{p}(\vec{x}'-\vec{x})}_{=\vec{p}(\vec{x}-\vec{x}')} - \underbrace{E_{\vec{p}}}_{=E_{-\vec{p}}} (t'-t)\right]}}_{\vec{p}' \to -\vec{p}} \right\}$$

$$= \int d^3p \sqrt{\frac{mc^2}{(2\pi\hbar)^2 \sqrt{\vec{p}^2c^2 + m^2c^4}}} \exp\left\{ \frac{i}{\hbar}\left[\vec{p}(\vec{x}-\vec{x}') - \sqrt{\vec{p}^2c^2+m^2c^4} \, |t - \underbrace{t'}_{t'>t}| \right] \right\} \quad (\cancel{t})$$

$$\underbrace{}_{=t'-t}$$

- further calculation: spherical coordinates for momenta
- technical details: lecture notes

$$G(\vec{x},t;\vec{x}',t') = \frac{i \left(\frac{mc}{\hbar}\right)^2}{4\pi \sqrt{c^2(t-t')^2 - (\vec{x}-\vec{x}')^2}} \underbrace{H_1^{(2)}}_{\text{Hankel function of second order}} \left(\frac{mc}{\hbar} \underbrace{\sqrt{c^2(t-t')^2 - (\vec{x}-\vec{x}')^2}}_{= g_{\mu\nu} x^\mu x^\nu} \right)$$

$$H_\nu^{(2)}(z) = J_\nu(z) - i N_\nu(z) \quad (8.405.2)$$

Bessel   von Neumann

particle mass enters only via Compton wave length $\lambda_C = 2\pi \frac{\hbar}{mc}$

Non-relativistic limit: $c \to \infty$

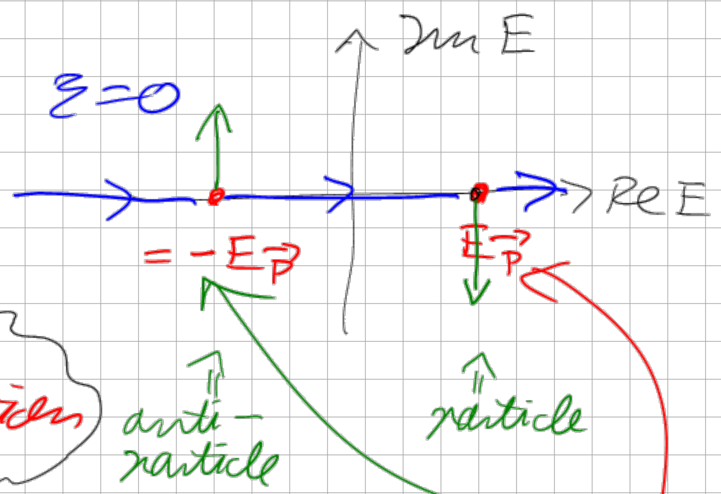
$$G(\vec{x},t;\vec{x}',t') \overset{c\to\infty}{=} \underbrace{\left(\frac{m}{2\pi i\hbar(t-t')}\right)^{\frac{3}{2}} e^{\frac{i}{\hbar}\frac{m(\vec{x}-\vec{x}')^2}{t-t'}}}_{=g(\vec{x},t;\vec{x}',t')} \cdot e^{-\frac{i}{\hbar}mc^2(t-t')}$$

4.13 Covariant Form of Propagator:

auxiliary integral:

$$I(t-t') = \boxed{\lim_{\varepsilon \downarrow 0}} \int_{-\infty}^{+\infty} \frac{dE}{2\pi\hbar} \frac{e^{-\frac{i}{\hbar}E(t-t')}}{E^2 - E_{\vec{p}}^2 \underbrace{+\, i\varepsilon}}$$

Feynman $i\varepsilon$-prescription

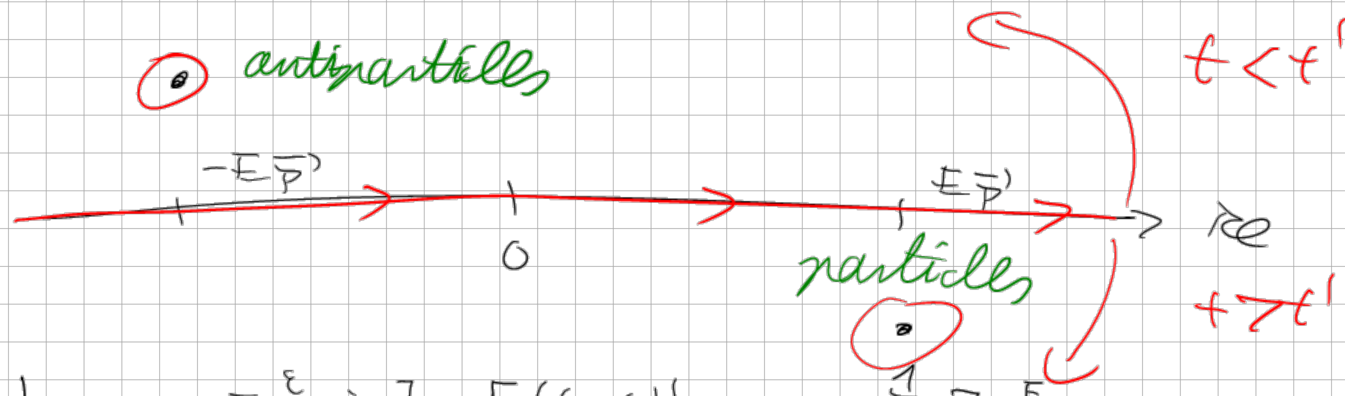


infinitesimal shift: $E = \pm\sqrt{E_{\vec{p}}^2 - i\varepsilon}$

$z = x + iy = r e^{i\varphi}$, $r = \sqrt{x^2+y^2}$, $\varphi = \arctan\frac{y}{x}$

$$z = E_{\vec{p}}^2 - i\varepsilon = \sqrt{E_{\vec{p}}^4 + \varepsilon^2}\; e^{-i\arctan\frac{\varepsilon}{E_{\vec{p}}^2}} \overset{\varepsilon\downarrow 0}{\approx} E_{\vec{p}}^2\, e^{-i\frac{\varepsilon}{E_{\vec{p}}^2}}$$

$$\sqrt{z} = \pm E_{\vec{p}}\, e^{-\frac{i}{2}\frac{\varepsilon}{E_{\vec{p}}^2}} \overset{\varepsilon\downarrow 0}{\approx} \pm E_{\vec{p}}\left(1 - i\frac{\varepsilon}{2E_{\vec{p}}^2}\right) = \pm\left(E_{\vec{p}} - i\frac{\varepsilon}{2E_{\vec{p}}}\right) = \pm E_{\vec{p}} \mp i\frac{\varepsilon}{2E_{\vec{p}}}$$



$$\left| e^{-\frac{i}{\hbar}E(t-t')} \right| = e^{-\frac{i}{\hbar} i \,\mathrm{Im}E(t-t')} = e^{\frac{1}{\hbar}\underbrace{\mathrm{Im}E(t-t')}_{<0}}$$

$\Rightarrow$ Residue theorem

$$\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\frac{e^{-\frac{i}{\hbar}E(t-t')}}{E^2-E_{\vec p}^2+i\varepsilon} = \frac{-i}{2\hbar E_{\vec p}}\boxed{e^{-\frac{i}{\hbar}E_{\vec p}|t-t'|}}$$

$$\Rightarrow \text{---} \Rightarrow G(x^\lambda,x'^\lambda) = 2i\hbar mc \lim_{\varepsilon\downarrow 0}\int\frac{d^4p}{(2\pi\hbar)^4}\,\frac{e^{-\frac{i}{\hbar}g_{\mu\nu}p^\mu(x^\nu-x'^\nu)}}{g_{\mu\nu}p^\mu p^\nu - m^2c^2+i\varepsilon}$$

manifestly covariant form of Klein-Gordon propagator

$$G(x^\lambda,x'^\lambda) = \int\frac{d^4p}{(2\pi\hbar)^4}\,\underbrace{G(p^\lambda)}\; e^{-\frac{i}{\hbar}g_{\mu\nu}p^\mu(x^\nu-x'^\nu)}$$

non-relativistic limit

$$g(\vec p,E) = \lim_{c\to\infty}\frac{1}{c}\,G(\vec p, E+mc^2) = \dots = \lim_{\varepsilon\downarrow 0}\frac{i\hbar}{E-\frac{\vec p^{\,2}}{2m}+i\varepsilon}$$