5 Maxwell Field:

Motivation:

- Electrodynamic processes $\hat{=}$ Maxwell theory
- Apparent contradiction:
  Maxwell equations $\hat{=}$ first quantized theory

  no Planck constant
- Resolution: massive extension of Maxwell theory
  derivatives $\hat{=}$ inverse Compton wave length $\frac{1}{\lambda_c} = \frac{mc}{\hbar}$
  limit $m \to 0 \Rightarrow \hbar$ disappears

Plan of this chapter:

- Relativistic covariant formulation of Maxwell theory
- Canonical field quantization $\hat{=}$ 2nd quantization
- Main obstackle: vanishing rest mass implies gauge symmetry
- Result: properties of photon like energy, momentum, spin
  2 helicities = phys. deg. of freedom for massless particles
- Counting: Flieβbach

	degrees of freedom	gauge degrees	counting
Maxwell	4 (A^μ)	1	$4 - 2 \cdot 1 = 2$
linearized Einstein	10 ($g_{\mu\nu} = g_{\nu\mu}$)	4	$10 - 2 \cdot 4 = 2$

number of phys. degrees of freedom

- Photon propagator = building block of QED

8.1 Maxwell Equations:

- Forces upon charges at rest or moving
  electric field $\vec{E}$, magnetic induction $\vec{B}$
- Origin: charge density ρ, current density $\vec{j}$
- Mathematically: James Clark Maxwell
- Basis: Helmholtz vector decomposition theorem

	electric field	magnetic field
homogeneous	$\operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday) (1)	$\operatorname{div} \vec{B} = 0$ (3)
inhomogeneous	$\operatorname{div} \vec{E} = \frac{1}{\varepsilon_0} \rho$ (Gauß) (2)	$\operatorname{rot} \vec{B} = \mu_0 \vec{j}$ (Oersted) $+ \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ (Maxwell) (4)

light velocity: $c = \frac{1}{\sqrt{\varepsilon_0 \cdot \mu_0}}$ (5)

- ρ and $\vec{j}$ are not independent:

$$\frac{\partial \rho}{\partial t} \overset{(2)}{=} \varepsilon_0 \,\mathrm{div}\, \frac{\partial \vec{E}}{\partial t} \overset{(4)}{=} \varepsilon_0 \left\{ c^2 \underbrace{\mathrm{div}\,\mathrm{rot}\,\vec{B}}_{=0} - \underbrace{c^2 \mu_0}_{\overset{(5)}{=} \frac{1}{\varepsilon_0}} \mathrm{div}\,\vec{j} \right\}$$

$\Rightarrow$ charge conservation: $\frac{\partial \rho}{\partial t} + \mathrm{div}\,\vec{j} = 0$

- SI: Système international d'unités
- QFT: Lorentz-Heaviside: $\varepsilon_0 = \mu_0 = c = 1$

5.2 Local Gauge Symmetry:

$\mathrm{div}\,\vec{B} \overset{(3)}{=} 0 \Rightarrow \boxed{\vec{B} = \mathrm{rot}\,\underbrace{\vec{A}}_{\text{vector potential}}}$ (6)

(6) in (1): $\mathrm{rot}\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = \vec{0} \Rightarrow \boxed{\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \mathrm{grad}\,\underbrace{\varphi}_{\text{scalar potential}}}$ (7)

(6), (7) + (2), (4):

$$\boxed{\frac{1}{\varepsilon_0}\rho \overset{(2)}{=} \mathrm{div}\,\vec{E} \overset{(7)}{=} -\underbrace{\mathrm{div}\,\mathrm{grad}\,\varphi}_{=\Delta\varphi} - \frac{\partial}{\partial t}\mathrm{div}\,\vec{A}} \quad (8)$$

$$\mu_0 \vec{j} \overset{(4)}{=} \mathrm{rot}\,\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \overset{(6),(7)}{=} \underbrace{\mathrm{rot}\,\mathrm{rot}\,\vec{A}}_{\mathrm{grad}\,\mathrm{div} - \Delta} - \frac{1}{c^2}\left(-\mathrm{grad}\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \vec{A}}{\partial t^2}\right)$$

$$\boxed{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \Delta \vec{A} + \mathrm{grad}\left(\boxed{\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \mathrm{div}\,\vec{A}}\right) = \mu_0 \vec{j}} \quad (9)$$

$\Rightarrow$ coupled differential equations: $\varphi, \vec{A}$

- local gauge symmetry:

$$\varphi' = \varphi + \frac{\partial \Lambda}{\partial t}, \qquad \vec{A}' = \vec{A} - \operatorname{grad}\Lambda$$

gauge function chosen arbitrarily

$$\vec{B}' = \operatorname{rot}\vec{A}' = \operatorname{rot}\vec{A} - \operatorname{rot}\operatorname{grad}\Lambda = \operatorname{rot}\vec{A} = \vec{B}$$

$$\vec{E}' = -\frac{\partial \vec{A}'}{\partial t} - \operatorname{grad}\varphi' = -\frac{\partial \vec{A}}{\partial t} + \cancel{\operatorname{grad}\frac{\partial \Lambda}{\partial t}} - \operatorname{grad}\varphi - \cancel{\operatorname{grad}\frac{\partial \Lambda}{\partial t}} = \vec{E}$$

(8) and (9): do also not change

- choosing a particular gauge, i.e. choosing a particular Λ, allows to decouple (8) and (9)

Coulomb gauge: $\operatorname{div}\vec{A} = 0$ ("longitudinal part of $\vec{A}$ vanishes")

$\overset{(8)}{\Rightarrow}$ $\Delta\varphi = -\frac{1}{\varepsilon_0}\rho$ (Poisson equation)

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \Delta\vec{A} = \mu_0\vec{j} - \frac{1}{c^2}\frac{\partial}{\partial t}\operatorname{grad}\varphi$$

$$\varphi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$$

φ is no longer a dynamical degree of freedom

$\Rightarrow$ Only 2 transversal degrees of freedom of $\vec{A}$ remain.

Lorentz gauge: $\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \operatorname{div} \vec{A} = 0$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi = \frac{1}{\varepsilon_0}\rho$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \Delta \vec{A} = \mu_0 \vec{j}$$

manifestly covariant

Disadvantage:

- Quantization: Gupta / Bleuler
- An unphysical longitudinal degree of freedom for $\vec{A}$ appears

5.3 Field Strength Tensor:

$\vec{E}, \vec{B} \Rightarrow$ antisymmetric 4×4 tensor $\quad F^{\mu\nu} = -F^{\nu\mu}$

$$\left(F^{\mu\nu}(\vec{E}, \vec{B})\right) = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

tensor of 2nd rank

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} F^{\mu'\nu'}$$

Dual electromagnetic field strength tensor

$${}^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\kappa} F_{\lambda\kappa}$$

$$F_{\lambda\kappa} = g_{\lambda\mu} g_{\kappa\nu} F^{\mu\nu}$$

$$(*F^{\mu\nu}) = \left(F^{\mu\nu}\left(+c\vec{B}, -\frac{\vec{E}}{c}\right)\right) = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$$

Concise summary of Maxwell equations:

homogeneous | inhomogeneous
$$\partial_\mu {}^*F^{\mu\nu} = 0 \qquad \partial_\mu F^{\mu\nu} = \mu_0 j^\nu$$

contravariant current density: $(j^\nu) = \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}$

Consistency equation:

$$\underbrace{\partial_\nu \partial_\mu}_{\text{symm. in } \mu,\nu} \underbrace{F^{\mu\nu}}_{\text{antisymm. in } \mu,\nu} = 0 = \mu_0 j^\nu \Rightarrow \partial_\nu j^\nu = 0 \hat{=} \frac{1}{c}\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0$$

5.4 Four Vector Potential:

$$(A^\lambda) = \left(\frac{\varphi}{c}, \vec{A}\right)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (\partial^\mu) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right) = \left(\frac{\partial}{\partial x_\mu}\right)$$

$$F^{01} = \partial^0 A^1 - \underbrace{\partial^1}_{= -\frac{\partial}{\partial x}} A^0 = \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{1}{c} = -\frac{1}{c}E_x$$

$$\partial_\mu {}^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\kappa}\partial_\mu \underbrace{F_{\lambda\kappa}}_{=\partial_\lambda A_\kappa - \partial_\kappa A_\lambda} = \dots = \frac{1}{2}\varepsilon^{\mu\nu\lambda\kappa}\underbrace{(\partial_\mu\partial_\lambda - \partial_\lambda\partial_\mu)}_{=0} A_\kappa \equiv 0$$

$\Rightarrow$ hom. Maxwell eq. fulfilled

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \mu_0 j^\nu \hat{=} (8),(9)$$

local gauge symmetry: $A'^\mu = A^\mu + \partial^\mu \Lambda \hat{=}$ above eq.

$$F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu = \partial^\mu A^\nu + \cancel{\partial^\mu \partial^\nu \Lambda} - \partial^\nu A^\mu - \cancel{\partial^\nu \partial^\mu \Lambda} = F^{\mu\nu}$$

$${}^*F'^{\mu\nu} = {}^*F^{\mu\nu}$$

5.5 Euler-Lagrange Equations:

$$S[A_\nu(\cdot)] = \frac{1}{c}\int d^4x\, \mathcal{L}, \quad \mathcal{L} = \mathcal{L}(A_\nu(x^\lambda), \partial_\mu A_\nu(x^\lambda))$$

$$\mathcal{L} = \alpha F^{\lambda\kappa}F_{\lambda\kappa} + \beta j^\lambda A_\lambda$$

$$\frac{\delta S}{\delta A_\nu(x^\lambda)} = \frac{\partial \mathcal{L}}{\partial A_\nu(x^\lambda)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu(x^\lambda))} = 0$$

$$\Rightarrow \underset{\text{sheet 5}}{\dots} \Rightarrow \partial_\mu F^{\mu\nu} = \frac{\beta}{4\alpha} j^\nu \overset{!}{=} \mu_0 j^\nu \Rightarrow \beta = 4\alpha\mu_0$$

$$\Rightarrow \mathcal{L} = \alpha F^{\mu\nu}F_{\mu\nu} + 4\alpha\mu_0 j^\mu A_\mu$$

5.6 Hamilton Function:

free electromagnetic field: $\rho = 0$, $\vec{j} = \vec{0}$

Coulomb gauge: $\operatorname{div} \vec{A} = 0$

$$\varphi(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|} \equiv 0$$

radiation gauge: $\varphi = 0$ and $\varphi \equiv 0$

from above: $\frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \Delta \vec{A} = 0$

$\vec{B} = \operatorname{rot} \vec{A}$, $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$

$$\mathcal{L} = \alpha F^{\mu\nu} F_{\mu\nu} = 2\alpha \left\{ (\vec{\nabla} \times \vec{A})^2 - \frac{1}{c^2} \left(\frac{\partial \vec{A}}{\partial t} \right)^2 \right\}$$

momentum field $\vec{\pi}(\vec{x}, t)$ canonically conjugated to $\vec{A}$

$$\vec{\pi} = \frac{\delta \mathcal{L}}{\delta \left(\frac{\partial \vec{A}}{\partial t} \right)} = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \vec{A}}{\partial t} \right)} = -\frac{4\alpha}{c^2} \frac{\partial \vec{A}}{\partial t} \Rightarrow \frac{\partial \vec{A}}{\partial t} = -\frac{c^2}{4\alpha} \vec{\pi}$$

$$\mathcal{H} = \vec{\pi} \cdot \frac{\partial \vec{A}}{\partial t} - \mathcal{L} = \dots = -\frac{c^2}{8\alpha} \vec{\pi}^2 - 2\alpha (\vec{\nabla} \times \vec{A})^2$$

Legendre transformation

$$= \frac{\varepsilon_0}{2} \vec{E}^2 + \frac{1}{2\mu} \vec{B}^2 = \frac{1}{2\varepsilon_0} \frac{c^4}{16\alpha^2} \vec{\pi}^2 + \frac{1}{2\mu} (\vec{\nabla} \times \vec{A})^2$$

↑ electromagnetic field

$$\alpha = -\frac{1}{4\mu_0}$$

consequence: $\vec{\pi} = -\frac{4\alpha}{c^2}\frac{\partial \vec{A}}{\partial t} = \frac{1}{\mu_0}\varepsilon_0\mu_0\frac{\partial \vec{A}}{\partial t} \Rightarrow$ $\boxed{\vec{\pi} = \varepsilon_0 \frac{\partial \vec{A}}{\partial t}}$

$\vec{p} = m \frac{d\vec{x}}{dt}$

Hamilton function:

$$H = \int d^3x\, \mathcal{H} = \frac{1}{2}\int d^3x \left\{ \frac{1}{\varepsilon_0}\vec{\pi}^2(\vec{x},t) + \frac{1}{\mu_0}\left[\vec{\nabla}\times\vec{A}\right]^2 \right\}$$

$\frac{1}{\varepsilon_0}\vec{\pi}^2$ $\hat{=}$ electric field energy $\hat{=}$ kinetic

$\frac{1}{\mu_0}[\vec{\nabla}\times\vec{A}]^2$ $\hat{=}$ magnetic field energy potential

side calculation:

$$(\vec{\nabla}\times\vec{A})^2 = \underbrace{\varepsilon_{jkl}}_{=\varepsilon_{klj}}(\partial_k A_l)\underbrace{\varepsilon_{jmn}}_{=\varepsilon_{mnj}}(\partial_m A_n) = (\delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm})(\partial_k A_l)(\partial_m A_n)$$

$$= \underline{(\partial_k A_l)(\partial_k A_l)} - \underbrace{(\partial_k A_l)(\partial_l A_k)}_{\underbrace{\partial_k(A_l\partial_l A_k)}_{=0 \text{ under the integral}} - \underbrace{A_l \partial_l \partial_k A_k}_{=0 \text{ Coulomb gauge}}}$$

$$H = \frac{1}{2}\int d^3x \left\{ \frac{1}{\varepsilon_0}\pi_k(\vec{x},t)\pi_k(\vec{x},t) + \frac{1}{\varepsilon_0}\partial_k A_l(\vec{x},t)\,\partial_k A_l(\vec{x},t) \right\}$$

5.6 Canonical Field Quantization:

fields		operators
$A_j(\vec{x},t)$	$\xrightarrow[\text{quantization}]{\text{second}}$	$\hat{A}_j(\vec{x},t)$
$\pi_j(\vec{x},t)$		$\hat{\pi}_j(\vec{x},t)$

How to define equal time commutation relations?

1) Independence of $\hat{A}_j$, $\hat{\pi}_j$ field operators:

$$[\hat{A}_k(\vec{x},t), \hat{A}_{k'}(\vec{x}',t)]_- = 0 = [\hat{\pi}_k(\vec{x},t), \hat{\pi}_{k'}(\vec{x}',t)]_-$$

2) Problem for mixed commutator due to Coulomb gauge:

$$\partial_k A_k(\vec{x},t) = 0 \quad \Rightarrow \quad \underline{\partial_k \hat{A}_k(\vec{x},t) = 0}$$

First trial:

$$[\hat{A}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_- = i\hbar \, \delta_{kl} \, \delta(\vec{x}-\vec{x}')$$

wrong! Take on both side ∂_k

$$0 = [\partial_k \hat{A}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_- \neq i\hbar \delta_{kl} \partial_k \delta(\vec{x}-\vec{x}') = i\hbar \partial_l \delta(\vec{x}-\vec{x}')$$

Second trial:

$$[\hat{A}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_- = i\hbar \, \delta^T_{kl}(\vec{x}-\vec{x}')$$

$T \leftarrow$ transversal delta function

have to determine such that $\partial_k \delta^T_{kl}(\vec{x}-\vec{x}') = 0$

Next time: "derivation", heuristic

Warning: All predictions following from this have to be

checked by experiments in quantum optics