# 5.7 Canonical Field Quantization:

fields $\qquad\qquad\qquad\qquad\qquad$ operators

$$\begin{matrix} A_j(\vec{x},t) & \xrightarrow[\text{quantization}]{\text{2nd}} & \hat{A}_j(\vec{x},t) \\ \pi_j(\vec{x},t) & & \hat{\pi}_j(\vec{x},t) \end{matrix}$$

Equal-time commutation relations:

$$[\hat{A}_j(\vec{x},t), \hat{A}_k(\vec{x}',t)]_- = 0 = [\hat{\pi}_j(\vec{x},t), \hat{\pi}_j(\vec{x}',t)]_-$$

non-trivial: $\quad [\hat{A}_j(\vec{x},t), \hat{\pi}_k(\vec{x}',t)]_- \neq i\hbar\, \delta_{jk}\, \delta(\vec{x}-\vec{x}')$

not valid due to $\partial_k \hat{A}_k(\vec{x},t) = 0$ transversality condition

transversal

instead: $\quad [\hat{A}_j(\vec{x},t), \hat{\pi}_k(\vec{x}',t)]_- = i\hbar\, \delta^{T}_{jk}(\vec{x}-\vec{x}')$

$$\Rightarrow \quad \partial_j \underbrace{\delta^{T}_{jk}(\vec{x}-\vec{x}')}_{\text{has to be determined}} = 0 \qquad \text{necessary condition} \qquad (*)$$

$$\delta^{T}_{jk}(\vec{x}-\vec{x}') = \int \frac{d^3k}{(2\pi)^3}\, \delta^{T}_{jk}(\vec{k})\, e^{i\vec{k}(\vec{x}-\vec{x}')}$$

$$(*): \quad \int \frac{d^3k}{(2\pi)^3}\, i k_j\, \delta^{T}_{jk}(\vec{k})\, e^{i\vec{k}(\vec{x}-\vec{x}')} = 0 \quad \Rightarrow \quad k_j\, \delta^{T}_{jk}(\vec{k}) = 0 \quad (**)$$

Solution ansatz: $\delta^T_{jk}(\vec{k}) = \delta_{jk} + \underbrace{\Delta_{jk}(\vec{k})}_{= k_j k_k \, f(\vec{k})}$

$(**)$: $k_k + \vec{k}^2 f(\vec{k})\, k_k = k_k \left(1 + \vec{k}^2 f(\vec{k})\right) = 0$

$\Rightarrow f(\vec{k}) = -\frac{1}{\vec{k}^2} \Rightarrow \delta^T_{jk}(\vec{k}) = \delta_{jk} - \frac{k_j k_k}{|\vec{k}|^2}$

$$\delta^T_{jk}(\vec{x}-\vec{x}') = \delta_{jk} \underbrace{\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}(\vec{x}-\vec{x}')}}_{=\delta(\vec{x}-\vec{x}')} - \overbrace{\int \frac{d^3k}{(2\pi)^3} \frac{1}{\vec{k}^2} \underbrace{k_j k_k}_{i\partial_j \; i\partial_k} e^{i\vec{k}(\vec{x}-\vec{x}')}}$$

$$= + \partial_j \partial_k \underbrace{\int \frac{d^3k}{(2\pi)^3} \frac{1}{\vec{k}^2} e^{i\vec{k}(\vec{x}-\vec{x}')}}_{= \frac{1}{|\vec{x}'-\vec{x}|} \frac{1}{4\pi}}$$

$$\delta^T_{jk}(\vec{x}-\vec{x}') = \delta_{jk}\, \delta(\vec{x}-\vec{x}') + \frac{1}{4\pi} \partial_j \partial_k \frac{1}{|\vec{x}-\vec{x}'|}$$

$$\left[\hat{A}_j(\vec{x},t), \hat{\pi}_k(\vec{x}',t)\right]_- = i\hbar\, \delta^T_{jk}(\vec{x}-\vec{x}')$$

Note: These commutations have to be checked by comparing their consequences with experimental results.

## 5.8 Heisenberg Equations:

Hamilton operator: $\hat{H} = \frac{1}{2}\int d^3x \left\{ \frac{1}{\varepsilon_0} \hat{\pi}_k(\vec{x},t)\, \hat{\pi}_k(\vec{x},t) + \frac{1}{\mu_0} \partial_k \hat{A}_l(\vec{x},t)\, \partial_k \hat{A}_l(\vec{x},t) \right\}$

Remark: operator ordering no problem here

$$i\hbar \frac{\partial}{\partial t} \hat{A}_j(\vec{x},t) = [\hat{A}_j(\vec{x},t), \hat{H}]_- \underset{\uparrow}{=} \frac{i\hbar}{\varepsilon_0} \int d^3x'\, \delta^T_{jk}(\vec{x}-\vec{x}')\, \hat{\pi}_k(\vec{x}',t)$$

abc rule, commutation relations

$$= \frac{i\hbar}{\varepsilon_0} \left\{ \hat{\pi}_j(\vec{x},t) + \cancel{\frac{1}{4\pi} \int d^3x' \left( \partial_j \partial_k \frac{1}{|\vec{x}-\vec{x}'|} \right) \underbrace{\partial'_k \hat{\pi}_k(\vec{x}',t)}_{= \varepsilon_0 \frac{\partial}{\partial t} \underbrace{\partial'_k \hat{A}_k(\vec{x}',t)}_{=0}}} \right\}$$ partial integration

$$\hat{\vec{\pi}}(\vec{x},t) = \varepsilon_0 \frac{\partial \hat{\vec{A}}(\vec{x},t)}{\partial t} \quad (1)$$

$$i\hbar \frac{\partial}{\partial t} \hat{\pi}_j(\vec{x},t) = \text{similar} = -\frac{i\hbar}{\mu_0} \Delta \hat{A}_j(\vec{x},t) \Rightarrow \frac{\partial \hat{\vec{\pi}}}{\partial t} = -\frac{1}{\mu_0} \Delta \hat{\vec{A}}(\vec{x},t) \quad (2)$$

(1) and (2): $\left( \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \right) \hat{\vec{A}}(\vec{x},t) = 0, \quad c^2 = \frac{1}{\varepsilon_0 \mu_0}$

wave equation

## 5.8 Decomposition in Plane Waves:

$$\hat{\vec{A}}(\vec{x},t) = \int d^3k\, \hat{\vec{A}}(\vec{k},t)\, e^{i\vec{k}\vec{x}} \underset{\text{equation}}{\overset{\text{wave}}{\Longrightarrow}} \left( \frac{\partial^2}{\partial t^2} + \omega_k^2 \right) \hat{\vec{A}}(\vec{k},t) = 0$$

$\omega_{\vec{k}} = c|\vec{k}| = \omega_{-\vec{k}}$

$$\hat{\vec{A}}(\vec{k},t) = \hat{\vec{A}}^{(1)}(\vec{k})\, e^{-i\omega_{\vec{k}} t} + \hat{\vec{A}}^{(2)}(\vec{k})\, e^{+i\omega_{\vec{k}} t}$$

$$\hat{\vec{A}}(\vec{x},t) = \int d^3k \left\{ \hat{\vec{A}}^{(1)}(\vec{k})\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}} t)} + \hat{\vec{A}}^{(2)}(\vec{k})\, e^{i(\vec{k}\vec{x}+\omega_{\vec{k}} t)} \right\}$$

$$\hat{\vec{A}}^{(2)}(\vec{k})\, e^{i(\vec{k}\vec{x}+\omega_{\vec{k}} t)} \xrightarrow{\vec{k}\to-\vec{k}} \hat{\vec{A}}^{(2)}(-\vec{k})\, e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}} t)}$$

electrically neutral field: $\hat{\vec{A}}(\vec{x},t) = \hat{\vec{A}}^{+}(\vec{x},t)$

$$\hat{\vec{A}}^{+}(\vec{x},t) = \int d^3k \left\{ \hat{\vec{A}}^{(2)+}(-\vec{k})\, e^{+i(\vec{k}\vec{x}-\omega_{\vec{k}} t)} + \hat{\vec{A}}^{(1)+}(\vec{k})\, e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}} t)} \right\}$$

$$\hat{\vec{A}}^{(1)}(\vec{k}) = \hat{\vec{A}}(\vec{k}), \quad \hat{\vec{A}}^{(2)}(\vec{k}) = \hat{\vec{A}}^{+}(-\vec{k})$$

$$\Rightarrow \hat{\vec{A}}(\vec{x},t) = \int d^3k \left\{ \hat{\vec{A}}(\vec{k})\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}} t)} + \hat{\vec{A}}^{+}(\vec{k})\, e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}} t)} \right\}$$

## 5.9 Construction of Polarization Vectors:

Note: here only classical

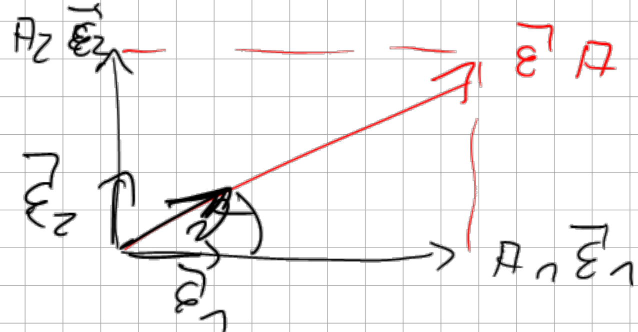
two linearly polarized plane waves

$$\vec{A}_1(\vec{x},t) = \underbrace{A_1}_{\text{amplitude}}\, \underbrace{\vec{\varepsilon}_1}_{\text{polarization vector}}\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}} t)}, \quad \hat{\vec{A}}_2(\vec{x},t) = A_2 \vec{\varepsilon}_2\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}} t)}$$

$$\vec{\varepsilon}_1^{\,*}\cdot\vec{\varepsilon}_1 = 1 = \vec{\varepsilon}_2^{\,*}\cdot\vec{\varepsilon}_2, \quad \vec{\varepsilon}_1^{\,*}\cdot\vec{\varepsilon}_2 = 0$$

$$\vec{A}(\vec{x},t) = \vec{A}_1(\vec{x},t) + \vec{A}_2(\vec{x},t) = \underbrace{(A_1\vec{\varepsilon}_1 + A_2\vec{\varepsilon}_2)}_{=A\vec{\varepsilon}}\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}} t)}$$

$$A = \sqrt{|A_1|^2+|A_2|^2}\; e^{i\vartheta}, \quad \vartheta = \arctan\frac{|A_2|}{|A_1|}$$




General case: $A_1 = |A_1| e^{i\varphi_1}$, $A_2 = |A_2| e^{i\varphi_2}$

$\varphi_1, \varphi_2$ arbitrary → elliptical polarization

here: circular polarization

$A_1 = \frac{A_0}{\sqrt{2}}$, $A_2 = \pm i \frac{A_0}{\sqrt{2}}$ : same amplitude, but phase shifted 90°

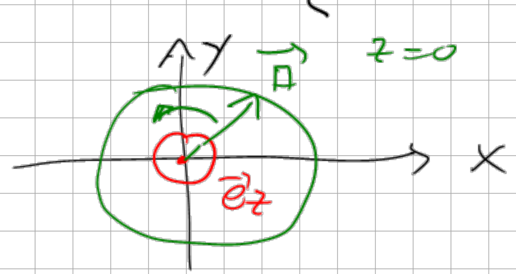
$$\Rightarrow \vec A(\vec x,t) = \frac{A_0}{\sqrt{2}}\left(\vec\varepsilon_1 \pm i\,\vec\varepsilon_2\right) e^{i(\vec k\vec x - \omega_k t)}$$

simplify: $\vec k = k\,\vec e_z$

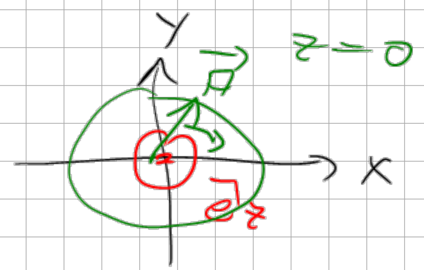
$$\vec k = \begin{pmatrix}0\\0\\1\end{pmatrix}k, \quad \vec\varepsilon_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \vec\varepsilon_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\Rightarrow \vec A(\vec x,t) = \underbrace{\frac{A_0}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\\ 0\end{pmatrix}}_{=\vec\varepsilon(k\vec e_z,\lambda)} e^{i(k\vec e_z\vec x - \omega_{k\vec e_z} t)}$$

$$\mathrm{Re}\,\vec A(\vec x,t) = \frac{A_0}{\sqrt{2}}\begin{pmatrix}\cos(kz-\omega_{k\vec e_z}t)\\ \mp\sin(kz-\omega_{k\vec e_z}t)\\ 0\end{pmatrix}$$




− sign
anticlockwise




+ sign
clockwise

optics    left-circular    right-circular

elementary particle physics    positive helicity    negative helicity

helicity: Chapter 2  $\hat h(\vec k) = \frac{\vec k}{|\vec k|}\cdot\vec S$, $\vec S = \begin{pmatrix} N^{23}\\ N^{31}\\ N^{12}\end{pmatrix} \xleftarrow{N^{\alpha\beta}=L^{\alpha\beta}} \vec L$

$$\hat h(\vec k) = \frac{-i}{k}\left\{ k_x \overbrace{\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}}^{=L_1} + k_y \underbrace{\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}}_{=L_2} + k_z \underbrace{\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}}_{=L_3}\right\}$$

$$= \frac{i}{k}\begin{pmatrix}0&-k_z&k_y\\k_z&0&-k_x\\-k_y&k_x&0\end{pmatrix}$$

plane wave propagation with wave vector $\vec k$ and helicity $\lambda=\pm1$

$$\vec A(\vec x,t) = A\,\underbrace{\vec\varepsilon(\vec k,\lambda)}_{\text{eigenvector of helicity operator}}\,e^{i(\vec k\vec x-\omega_k t)}$$

$$\hat h(\vec k)\,\vec\varepsilon(\vec k,\lambda) = \lambda\,\vec\varepsilon(\vec k,\lambda)$$

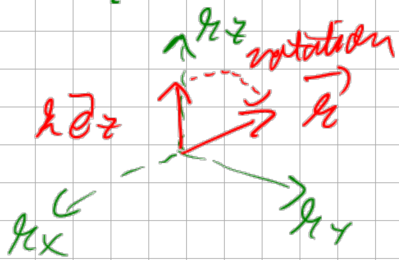
special case: $\vec\varepsilon(k\vec e_z,\lambda) = \frac{1}{\sqrt2}\begin{pmatrix}1\\ \lambda i\\ 0\end{pmatrix}$

$$\underbrace{i\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}}_{=\hat h(k\vec e_z)}\frac{1}{\sqrt2}\begin{pmatrix}1\\ \lambda i\\ 0\end{pmatrix} = \dots \underset{\lambda^2=1}{=} \lambda\frac{1}{\sqrt2}\begin{pmatrix}1\\ \lambda i\\ 0\end{pmatrix}$$

$$\hat{h}(k\vec{e}_z)\,\vec{\varepsilon}(k\vec{e}_z,\lambda) = \lambda\,\vec{\varepsilon}(k\vec{e}_z,\lambda)$$

How to construct from this $\vec{\varepsilon}(\vec{k},\lambda)$?

Notation: $k\vec{e}_z \xrightarrow{\text{rotation}} \vec{k} \Rightarrow \vec{\varepsilon}(k\vec{e}_z,\lambda) \xrightarrow[\text{rotation}]{\text{same}} \vec{\varepsilon}(\vec{k},\lambda)$




Rotation: $R(\theta,\phi) = R_z(\phi)\,R_y(\theta)$

$$R_z(\phi) = e^{-iL_3\phi} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\theta) = e^{-iL_2\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\Rightarrow R(\theta,\phi) = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\vec{k} = R(\theta,\phi)\,k\vec{e}_z = k\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$ $\vec{k}$ in spherical coordinates

$$\vec{\varepsilon}(\vec{k},\lambda) = R(\Theta,\phi)\,\vec{\varepsilon}(k\vec{e}_z,\lambda) = \ldots = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\Theta\cos\phi - \lambda i \sin\phi \\ \cos\Theta \sin\phi + \lambda i \cos\phi \\ -\sin\Theta \end{pmatrix}$$

$$\hat{h}(\vec{k})\,\vec{\varepsilon}(\vec{k},\lambda) = \ldots = \lambda\,\vec{\varepsilon}(\vec{k},\lambda)$$

$$\vec{\varepsilon}(\vec{k},\lambda)\Big|_{\substack{\Theta=0\\ \phi=0}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \lambda i \\ 0 \end{pmatrix} = \vec{\varepsilon}(k\vec{e}_z,\lambda)$$

## 5.10 Properties of Polarization Vectors:

$$\hat{\vec{A}}(\vec{x},t) = \int d^3k \left\{ \hat{\vec{A}}(\vec{k})\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)} + \hat{\vec{A}}^{+}(\vec{k})\, e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}}t)} \right\}$$

Coulomb gauge: $\operatorname{div}\vec{A}(\vec{x},t) \equiv 0 \implies \vec{k}\cdot\hat{\vec{A}}(\vec{k}) = 0$

transversality condition

polarization vectors: $\vec{k}\cdot\vec{\varepsilon}(\vec{k},\lambda) = 0$

$$\hat{\vec{A}}(\vec{k}) = \underbrace{N_{\vec{k}}}_{\text{normalization constant}} \sum_{\lambda=\pm 1} \vec{\varepsilon}(\vec{k},\lambda)\,\hat{a}_{\vec{k},\lambda}$$

Note 1: $\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{\varepsilon}(\vec{k},\lambda')^{*} = \delta_{\lambda\lambda'}$

Note 2: $\vec{k} \to -\vec{k}$

$\phi \to \phi + \pi$ : $\sin\phi \to -\sin\phi$, $\cos\phi \to -\cos\phi$

(1) $\to$ (1) $-\pi$: $\sin(1) \to \sin(1)$, $\cos(1) \to -\cos(1)$

$$\vec{\varepsilon}(-\vec{k},\lambda) = \dots = \vec{\varepsilon}(\vec{k},-\lambda) = \vec{\varepsilon}(\vec{k},\lambda)^*$$

Result:

$$\boxed{\hat{\vec{A}}(\vec{x},t) = \sum_{\lambda=\pm1} \int d^3k\, N_{\vec{k}} \left\{ \vec{\varepsilon}(\vec{k},\lambda)\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\, \underline{\hat{a}_{\vec{k},\lambda}} + \vec{\varepsilon}^*(\vec{k},\lambda)\, e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\, \underline{\hat{a}^+_{\vec{k},\lambda}} \right\}}$$

## 5.11 Fourier Operators:

$$\hat{\vec{\pi}}(\vec{x},t) = \varepsilon_0 \frac{\partial \hat{\vec{A}}(\vec{x},t)}{\partial t} = \dots$$

$$\hat{a}^{+}_{\vec{k},\lambda} = \frac{1}{2(2\pi)^3 N_{\vec{k}}} \int d^3x\, \vec{\varepsilon}^{\cancel{*}}(\vec{k},\lambda)\, e^{\pm i(\vec{k}\vec{x}-\omega_{\vec{k}}t)} \left\{ \hat{\vec{A}}(\vec{x},t) \pm i \frac{\hat{\vec{\pi}}(\vec{x},t)}{\varepsilon_0 \omega_{\vec{k}}} \right\}$$

$$[\hat{A}_k(\vec{x},t), \hat{A}_e(\vec{x}',t)]_- = 0 \qquad [\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}]_- = 0$$

$$[\hat{\pi}_k(\vec{x},t), \hat{\pi}_e(\vec{x}',t)]_- = 0 \quad \Longleftrightarrow \quad [\hat{a}^+_{\vec{k},\lambda}, \hat{a}^+_{\vec{k}',\lambda'}]_- = 0$$

$$[\hat{A}_k(\vec{x},t), \hat{\pi}_e(\vec{x}',t)]_- = i\hbar\, \delta^T_{ke}(\vec{x}-\vec{x}') \Leftrightarrow [\hat{a}_{\vec{k},\lambda}, \hat{a}^+_{\vec{k}',\lambda'}]_-$$

$$N_{\vec{k}} = \sqrt{\frac{\hbar}{\cancel{2(2\pi)^3}\, \varepsilon_0 \omega_{\vec{k}}}} \quad \Leftarrow \quad = \underbrace{\frac{\hbar}{2(2\pi)^3 \varepsilon_0 \omega_{\vec{k}} N_{\vec{k}}^2}}_{=1} \delta_{\lambda\lambda'}\, \underbrace{\delta(\vec{k}-\vec{k}')}_{\delta_{\vec{k},\vec{k}'}}$$

$V$

Conclusion: $\hat{a}_{\vec{k},\lambda}$ / $\hat{a}^{\dagger}_{\vec{k},\lambda}$ are annihilation/creation operators of bosonic particles with wave vector $\vec{k}$ and helicity $\lambda$ ($\equiv$ photons)

Current exercise sheet: Properties of photons

- energy $\hbar\omega_{\vec{k}}$ , $\omega_{\vec{k}} = c|\vec{k}|$
- momentum $\hbar\vec{k}$
- $\boxed{\text{spin} \quad \hbar\lambda \frac{\vec{k}}{k}}$

Noether theorem + 2nd quantization