Second quantized vector potential:

$$\hat{\vec{A}}(\vec{x},t) = \sum_{\lambda=\pm1} \int d^3k \sqrt{\frac{\hbar}{2(2\pi)^3 \varepsilon_0 \omega_{\vec{k}}}} \Big\{ \vec{\varepsilon}(\vec{k},\lambda)\, e^{i(\vec{k}\vec{x} - \omega_{\vec{k}} t)}\, \hat{a}_{\vec{k},\lambda}$$

$$+ \vec{\varepsilon}^{\,*}(\vec{k},\lambda)\, e^{-i(\vec{k}\vec{x} - \omega_{\vec{k}} t)}\, \hat{a}^{+}_{\vec{k},\lambda} \Big\}$$

$$[\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}]_- = 0 = [\hat{a}^{+}_{\vec{k},\lambda}, \hat{a}^{+}_{\vec{k}',\lambda'}]_-$$

$$[\hat{a}_{\vec{k},\lambda}, \hat{a}^{+}_{\vec{k}',\lambda'}]_- = \delta_{\lambda\lambda'}\, \delta(\vec{k}-\vec{k}')$$

$$\hat{\vec{\pi}}(\vec{x},t) = \varepsilon_0 \frac{\partial \hat{\vec{A}}(\vec{x},t)}{\partial t}$$

$\Rightarrow$ basis of quantum optics.

Now: energy, momentum, spin angular momentum

$\Rightarrow$ properties of photon

5.13 Energy:

$$H = \int d^3x \left\{ \frac{\varepsilon_0}{2} \vec{E}^2(\vec{x},t) + \frac{1}{2\mu_0} \vec{B}(\vec{x},t) \right\}$$

$$\frac{1}{2}\int d^3x \left\{ \pi_k(\vec{x},t)\,\pi_k(\vec{x},t)\,\frac{1}{\varepsilon_0} + \frac{1}{\mu_0}\,\partial_k A_l(\vec{x},t)\,\partial_k A_l(\vec{x},t) \right\}$$

$$\hat{H} = \int d^3x\, \frac{1}{2} \left\{ \frac{1}{\varepsilon_0}\hat{\pi}_k(\vec{x},t)\,\hat{\pi}_k(\vec{x},t) + \frac{1}{\mu_0}\,\partial_k \hat{A}_l(\vec{x},t)\,\partial_k \hat{A}_l(\vec{x},t) \right\}$$

Note: no operator ordering problem

⋮

$$\hat{H} = \frac{1}{2} \sum_{\lambda=\pm1} \int d^3k\, \hbar\omega_{\vec{k}} \left(\hat{a}^+_{\vec{k}\lambda}\,\hat{a}_{\vec{k}\lambda} + \hat{a}_{\vec{k}\lambda}\,\hat{a}^+_{\vec{k}\lambda} \right)$$

vacuum state: $|0\rangle$, $\quad \hat{a}_{\vec{k}\lambda}|0\rangle = 0, \quad \langle 0|\hat{a}^+_{\vec{k}\lambda} = 0$

$$\langle 0|\hat{H}|0\rangle = \frac{1}{2}\cdot 2 \int d^3k\, \hbar\omega_{\vec{k}} = \text{divergent}$$

renormalized Hamilton operator

$$:\hat{H}: = \hat{H} - \langle 0|\hat{H}|0\rangle = \sum_{\lambda=\pm1} \int d^3k\, \hbar\omega_{\vec{k}}\, \underbrace{\hat{a}^+_{\vec{k}\lambda}\,\hat{a}_{\vec{k}\lambda}}_{\text{number operator}}$$

is time independent

5.14 Momentum:

$\vec{P} = \int d^3x \, \frac{1}{c^2} \vec{S}(\vec{x},t)$, Poynting vector $\quad \vec{S}(\vec{x},t) = \frac{1}{\mu_0} \underbrace{\vec{E}(\vec{x},t)}_{= -\vec{\pi}(\vec{x},t)} \times \underbrace{\vec{B}(\vec{x},t)}_{= \vec{\nabla}\times\vec{A}(\vec{x},t)}$

$$\vec{P} = \int d^3x \left[\vec{\nabla} \times \vec{A}(\vec{x},t)\right] \times \vec{\pi}(\vec{x},t)\, \varepsilon_0$$

$$\hat{\vec{P}} = \int d^3x \left[\vec{\nabla} \times \hat{\vec{A}}(\vec{x},\underline{\underline{t}})\right] \times \varepsilon_0 \, \hat{\vec{\pi}}(\vec{x},\underline{\underline{t}})$$

time independent

Note: operator ordering is important

Inserting plane wave decomposition

$$\hat{\vec{P}} = \dots = \sum_{\lambda=\pm 1} \int d^3k \, \frac{\hbar \vec{k}}{2} \left(\hat{a}^+_{\vec{k}\lambda} \hat{a}_{\vec{k}\lambda} + \hat{a}_{\vec{k}\lambda} \hat{a}^+_{\vec{k}\lambda} \right)$$

$$\langle 0 | \hat{\vec{P}} | 0 \rangle = \frac{1}{2} \sum_\lambda \int d^3k \, \hbar \vec{k} = \vec{0}$$

$$\hat{\vec{P}} = :\hat{\vec{P}}: = \vec{P} - \langle 0 | \hat{\vec{P}} | 0 \rangle = \sum_{\lambda=\pm 1} \int d^3k \, \hbar \vec{k} \, \hat{a}^+_{\vec{k}\lambda} \hat{a}_{\vec{k}\lambda}$$

5.15 Spin Angular Momentum:

Result from Noether theorem

$$\vec{S} = \int d^3x \, \underbrace{\varepsilon_0 \vec{E}(\vec{x},t)}_{= -\vec{\pi}(\vec{x},t)} \times \vec{A}(\vec{x},t) = \int d^3x \, \vec{A}(\vec{x},t) \times \vec{\pi}(\vec{x},t)$$

analogous to classical physics $\quad \vec{L} = \vec{r} \times \vec{p}, \quad \vec{p} = m\vec{v}, \quad \vec{\pi} = \varepsilon_0 \frac{\partial \vec{A}}{\partial t}$

$$\hat{\vec{S}} = \int d^3x \, \hat{\vec{A}}(\vec{x},t) \times \hat{\vec{\pi}}(\vec{x},t)$$

Note: operator ordering is important

Plane wave decomposition and

$$\vec{\varepsilon}(\vec{k},\lambda) \times \vec{\varepsilon}(\vec{k},\lambda')^* = -i\lambda \frac{\vec{k}}{k} \delta_{\lambda\lambda'}$$

$$\hat{\vec{S}} = \dots = \sum_{\lambda=\pm1} \int d^3k \, \lambda \frac{\hbar}{2} \frac{\vec{k}}{k} \left(\hat{a}^\dagger_{\vec{k}\lambda} \hat{a}_{\vec{k}\lambda} + \hat{a}_{\vec{k}\lambda} \hat{a}^\dagger_{\vec{k}\lambda} \right)$$ time independent

$$\langle 0 | \hat{\vec{S}} | 0 \rangle = \frac{1}{2} \Big(\sum_{\lambda=\pm1} \lambda \Big) \Big(\int d^3k \, \frac{\vec{k}}{k} \Big) = \vec{0}$$

$$:\hat{\vec{S}}: = \hat{\vec{S}} = \hat{\vec{S}} - \langle 0 | \hat{\vec{S}} | 0 \rangle = \sum_{\lambda=\pm1} \int d^3k \, \lambda \hbar \frac{\vec{k}}{k} \hat{a}^\dagger_{\vec{k}\lambda} \hat{a}_{\vec{k}\lambda}$$

Result 1: Energy, momentum and spin angular momentum for the electromagnetic field are time independent in second quantization and, therefore, represent conserved quantities

Result 2: $\hat{a}_{\vec{k}\lambda}$, $\hat{a}^{\dagger}_{\vec{k}\lambda}$ are annihilation, creation operators of photons with energy $\hbar\omega_{\vec{k}}$, momentum $\hbar\vec{k}$, and spin angular momentum $\lambda\hbar\frac{\vec{k}}{k}$, i.e. helicity λ

5.16 Definition of Maxwell Propagator:

$$D^{\mu\nu}(\vec{x},t;\vec{x}',t') = \langle 0 | \underbrace{\hat{T}\left(\hat{A}^{\mu}(\vec{x},t)\hat{A}^{\nu}(\vec{x}',t')\right)} | 0 \rangle$$

$$= \Theta(t-t')\,\hat{A}^{\mu}(\vec{x},t)\,\hat{A}^{\nu}(\vec{x}',t') + \Theta(t'-t)\,\hat{A}^{\nu}(\vec{x}',t')\,\hat{A}^{\mu}(\vec{x},t)$$

bosonic time ordering

manifestly not covariant

radiation gauge: $\hat{A}^{0}(\vec{x},t) = 0$

$D^{\mu\nu}(\vec{x},t;\vec{x}',t') = 0$ if either $\mu = 0$ or $\nu = 0$ $\Rightarrow$ $(D^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & \boxed{D^{jk}} \end{pmatrix}$

$$\frac{\partial D^{jk}(\vec{x},t;\vec{x}',t')}{\partial t} = \delta(t-t')\langle 0|\underbrace{[\hat{A}^j(\vec{x},t),\ \hat{A}^k(\vec{x}',t')]_-}_{=0}|0\rangle$$

$$+\Theta(t-t')\langle 0|\frac{\partial\hat{A}^j(\vec{x},t)}{\partial t}\hat{A}_k(\vec{x}',t')|0\rangle+\Theta(t'-t)\langle 0|\hat{A}_k(\vec{x}',t')\frac{\partial\hat{A}_j(\vec{x},t)}{\partial t}|0\rangle$$

$$\frac{\partial^2 D^{jk}(\vec{x},t;\vec{x}',t')}{\partial t^2} = \delta(t-t')\langle 0|\underbrace{[\underbrace{\frac{\partial\hat{A}^j(\vec{x},t)}{\partial t}}_{=\frac{1}{\varepsilon_0}\hat{\Pi}^j(\vec{x},t)},\ \hat{A}^k(\vec{x}',t')]_-}_{=-\frac{1}{\varepsilon_0}i\hbar\,\delta^T_{jk}(\vec{x}-\vec{x}')}|0\rangle$$

$$+\Theta(t-t')\langle 0|\underbrace{\frac{\partial^2\hat{A}^j(\vec{x},t)}{\partial t^2}}_{=c^2\Delta\hat{A}^j(\vec{x},t)}\hat{A}_k(\vec{x}',t')|0\rangle+\Theta(t'-t)\langle 0|\hat{A}^k(\vec{x}',t')\underbrace{\frac{\partial^2\hat{A}^j(\vec{x},t)}{\partial t^2}}_{=c^2\Delta\hat{A}^j(\vec{x},t)}|0\rangle$$

$$\Rightarrow\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\underbrace{D^{jk}(\vec{x},t;\vec{x}',t')}_{\text{transversal Maxwell propagator}} = -i\hbar\mu_0\,\delta(t-t')\,\delta^T_{jk}(\vec{x}-\vec{x}')$$

transversal Maxwell propagator = Greens function of Maxwell equation

5.17 Calculation of Maxwell Propagator:

$$D_{jk}(\vec{x},t;\vec{x}',t') = i\,\Theta(t-t')\langle 0|\hat{A}_j(\vec{x},t)\hat{A}_k(\vec{x}',t')|0\rangle + \begin{pmatrix} \vec{x}\leftrightarrow\vec{x}',\ t\leftrightarrow t' \\ j\leftrightarrow k \end{pmatrix}$$

plane wave decomposition

$$= \sum_{\lambda=\pm1}\sum_{\lambda'=\pm1}\int d^3k\int d^3k' \sqrt{\frac{\hbar}{2(2\pi)^3\varepsilon_0\omega_{\vec{k}}}}\cdot\sqrt{\frac{\hbar}{2(2\pi)^3\varepsilon_0\omega_{\vec{k}'}}}\; i\,\Theta(t-t')$$

$$\langle 0|\left[\varepsilon_j(\vec{k},\lambda)e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\hat{a}_{\vec{k}\lambda} + \cancel{\varepsilon_j^*(\vec{k},\lambda)e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\hat{a}^+_{\vec{k}\lambda}}\right]$$

$$\cdot\left[\cancel{\varepsilon_k(\vec{k}',\lambda')e^{i(\vec{k}'\vec{x}'-\omega_{\vec{k}'}t')}\hat{a}_{\vec{k}'\lambda'}} + \varepsilon_k^*(\vec{k}',\lambda')e^{-i(\vec{k}'\vec{x}'-\omega_{\vec{k}'}t')}\hat{a}^+_{\vec{k}'\lambda'}\right]|0\rangle$$

$$+\quad t\leftrightarrow t',\quad \vec{x}\leftrightarrow\vec{x}',\quad j\leftrightarrow k$$

$$\langle 0|\hat{a}_{\vec{k}\lambda}\hat{a}^+_{\vec{k}'\lambda'}|0\rangle = \delta_{\lambda\lambda'}\,\delta(\vec{k}-\vec{k}') + \underbrace{\langle 0|\hat{a}^+_{\vec{k}'\lambda'}\hat{a}_{\vec{k}\lambda}|0\rangle}_{=0}$$

$$D^{jk}(\vec{x},t;\vec{x}',t') = \sum_{\lambda=\pm1} \int d^3k \, \frac{\hbar}{2(2\pi)^3 \varepsilon_0 \omega_{\vec{k}}} \Big\{ \Theta(t-t') \, \varepsilon_j(\vec{k},\lambda) \varepsilon_k^*(\vec{k},\lambda) \, e^{i[\vec{k}(\vec{x}-\vec{x}')-\omega_{\vec{k}}(t-t')]} + \Theta(t'-t) \, \underbrace{\varepsilon_k(\vec{k},\lambda)}_{\varepsilon_k(\vec{k},-\lambda)} \underbrace{\varepsilon_j^*(\vec{k},\lambda)}_{\varepsilon_j^*(\vec{k},-\lambda)} e^{-i[\vec{k}(\vec{x}-\vec{x}')-\omega_{\vec{k}}(t-t')]} \Big\}$$

$$\lambda \to -\lambda$$

$$\underbrace{\varepsilon_k(\vec{k},-\lambda)}_{=\varepsilon_k^*(\vec{k},\lambda)} \quad \underbrace{\varepsilon_j^*(\vec{k},-\lambda)}_{=\varepsilon_j(\vec{k},\lambda)}$$

$$\underbrace{D^{jk}(\vec{x},t;\vec{x}',t')}_{=D^{jk}(\vec{x}-\vec{x}';t-t')} = \int d^3k \, \frac{\hbar \, P^{jk}(\vec{k})}{2(2\pi)^3 \varepsilon_0 \omega_{\vec{k}}} \Big\{ \Theta(t-t') \, e^{i[\vec{k}(\vec{x}-\vec{x}')-\omega_{\vec{k}}(t-t')]} + \Theta(t'-t) \, e^{-i[-\vec{k}(\vec{x}-\vec{x}')-\omega_{\vec{k}}(t-t')]} \Big\}$$

$$\vec{k} \to -\vec{k}$$

polarization sum: $P^{jk}(\vec{k}) = \sum_{\lambda=\pm1} \varepsilon_j(\vec{k},\lambda)\varepsilon_k^*(\vec{k},\lambda)$

$$P^{jk}(-\vec{k}) = \sum_{\lambda=\pm1} \underbrace{\varepsilon_j(-\vec{k},\lambda)}_{=\varepsilon_j(\vec{k},-\lambda)} \underbrace{\varepsilon_k^*(-\vec{k},\lambda)}_{=\varepsilon_k^*(\vec{k},-\lambda)} \overset{\lambda=-\lambda}{=\!=} P^{jk}(\vec{k})$$

$$D_{jk}(\vec{x},t;\vec{x}',t') = \int d^3k \, \frac{4\pi \, P_{jk}(\vec{k})}{2(2\pi)^3 \, \varepsilon_0 \, 2\omega_{\vec{k}}} \, e^{i\vec{k}(\vec{x}-\vec{x}')}$$

$$\left\{ \Theta(t-t') \, e^{-i\omega_{\vec{k}}(t-t')} + \Theta(t'-t) \, e^{i\omega_{\vec{k}}(t-t')} \right\}$$

$$P_{jk}(\vec{k}) = \sum_{\lambda=\pm1} \varepsilon_j(\vec{k},\lambda) \, \varepsilon_k^*(\vec{k},\lambda)$$

$$\vec{k} = k \begin{pmatrix} \sin\Theta \cos\phi \\ \sin\Theta \sin\phi \\ \cos\Theta \end{pmatrix}, \quad \vec{\varepsilon}(\vec{k},\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\Theta\cos\phi - \lambda i \sin\phi \\ \cos\Theta \sin\phi + \lambda i \cos\phi \\ -\sin\Theta \end{pmatrix}$$

$$= \ldots\ldots = \begin{pmatrix} 1-\frac{k_x^2}{k^2} & & \text{symm} \\ -\frac{k_x k_y}{k^2} & 1-\frac{k_y^2}{k^2} & \\ -\frac{k_x k_z}{k^2} & -\frac{k_y k_z}{k^2} & 1-\frac{k_z^2}{k^2} \end{pmatrix} = \left(\delta_{jk} - \frac{k_j k_k}{k^2} \right)$$

transversality condition: $k_k P_{jk}(k) = k_j - k_j \frac{k^2}{k^2} \equiv 0$

$\rightarrow \partial_j D_{jk}(\vec{x},t;\vec{x}',t') = 0$ transversal Maxwell propagator

5.18 Four-dimensional Fourier representation:

integral identity (see Klein-Gordon chapter):

$$\lim_{\xi\downarrow 0}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega^2-\omega_{\vec{k}}^2+i\xi}=\frac{-i}{2\omega_k}\left\{\Theta(t-t')e^{-i\omega_{\vec{k}}(t-t')}+(t\leftrightarrow t')\right\}$$

$$D^{jk}(\vec{x},t;\vec{x}',t')=\lim_{\xi\downarrow 0}\int\frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{i\hbar}{\epsilon_0}P^{jk}(\vec{k})\frac{e^{i[\vec{k}(\vec{x}-\vec{x}')-\omega_{\vec{k}}(t-t')]}}{\omega^2-\omega_{\vec{k}}^2+i\xi}$$

Note: not covariant

Aim: decompose this into a Lorentz invariant and a not Lorentz invariant contribution

$$(k^\lambda)=\begin{pmatrix}\omega/c\\ \vec{k}\end{pmatrix},\quad d^4k=\frac{1}{c}d^3k\,d\omega,\quad \omega^2-\omega_{\vec{k}}^2=c^2k^\lambda k_\lambda$$

$$D^{\mu\nu}(\vec{x},t;\vec{x}',t')=\lim_{\xi\downarrow 0}\frac{i\hbar}{c\,\epsilon_0}\int\frac{d^4k}{(2\pi)^4}\frac{P'^{\mu\nu}(k^\lambda)}{k^\lambda k_\lambda+i\xi}e^{ik_\lambda(x^\lambda-x'^\lambda)}$$

polarization sum

$$(P^{\mu\nu}(k^\lambda)) = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \underbrace{P^{ij}(\vec{k})}_{1 - \frac{k^i k^j}{k^2}} & \\ 0 & & & \end{array}\right) = \underbrace{(-g^{\mu\nu})}_{\text{covariant}} + \underbrace{\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & -\frac{k^i k^j}{k^2} & \\ 0 & & & \end{array}\right)}_{\text{not covariant}}$$

Aim: investigate further not covariant contribution

time-like vector: $(\xi^\lambda) = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}$

space-like vector: $(\bar{k}^\lambda) = \begin{pmatrix} 0 \\ \frac{\vec{k}}{|\vec{k}|} \end{pmatrix}$

$$\xi^\lambda \bar{k}_\lambda = 0$$

$k^\lambda \xi_\lambda = k^0 = \frac{\omega}{c}$, $k^2 = \left(\frac{\omega}{c}\right)^2 - \vec{k}^2$, $\sqrt{(k\xi)^2 - k^2} = |\vec{k}|$

$$(k^\lambda - (k\xi)\xi^\lambda) = \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix} - \frac{\omega}{c}\begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vec{k} \end{pmatrix} \Rightarrow \bar{k}^\lambda = \frac{k^\lambda - (k\xi)\xi^\lambda}{\sqrt{(k\xi)^2 - k^2}}$$

$$P^{\mu\nu} = \dots = \boxed{-g^{\mu\nu}} \; \boxed{- k^2 \frac{\xi^\mu \xi^\nu}{(k\xi)^2 - k^2}} \; \boxed{- \frac{k^\mu k^\nu - (k\xi)(k^\mu \xi^\nu + k^\nu \xi^\mu)}{(k\xi)^2 - k^2}}$$

$$D^{\mu\nu}(x;x') = \underbrace{D_F^{\mu\nu}(x;x')}_{\text{Feynman}} - \underbrace{D_C^{\mu\nu}(x;x')}_{\text{coulomb}} - \underbrace{D_R^{\mu\nu}(x;x')}_{\text{Residual}}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{\hbar}{c\varepsilon_0} \int \frac{d^4k}{(2\pi)^4} \frac{-i g^{\mu\nu}}{k^2 + i\varepsilon} e^{ik(x-x')}$$

- covariant
- obtained Gupta-Bleuler quantiz.
- yields physical results

- not covariant
- do not contribute to physical results

$$D_C^{\mu\nu}(x;x') = \frac{i\hbar\mu_0}{4\pi} \delta^{\mu 0} \delta^{\nu 0} \frac{\delta(t-t')}{|\vec{x}-\vec{x}'|}$$

- couples to charge density
- turns out to cancel with other terms

$$\partial_\mu j^\mu(x) = 0 \Rightarrow k_\mu j^\mu(k) = 0$$

$$\int \frac{d^4k}{(2\pi)^4} j_\mu(-k) D_R^{\mu\nu}(k) j_\nu(k) = 0$$

vanishes due to charge conservation