Chapter 6: Dirac

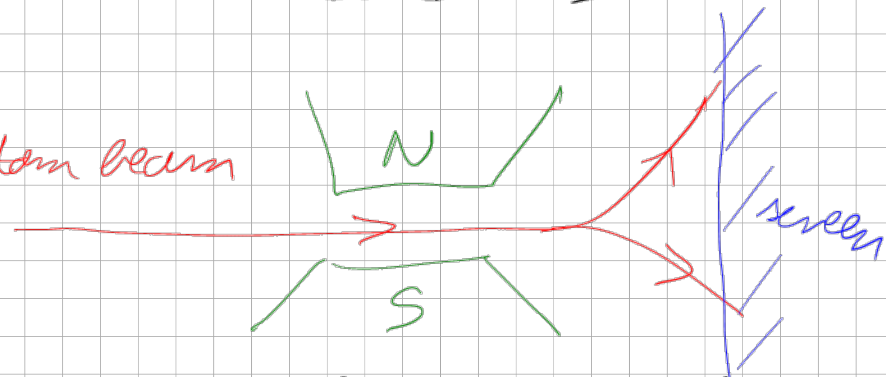
Motivation:

- Paul Dirac 1928
  - unification of special relativity and quantum mechanics
    Dirac equation = relativistic wave equation
  - spin 1/2 particles like electron/positron, quarks
  - validation: accounts for fine details of hydrogen atom spectrum
  - implies existence of antimatter, not known at that time;
    1932: detection of positron by Anderson
- Dirac equation for 4 complex fields = spinor
  - spinors transform differently than vectors with respect to rotations, boosts:
  - e.g. one has to rotate 720° in real space in order to reproduce the original spinor
  - non-relativistic limit: Pauli two-component wave function
  - massless spin 1/2 particles: Weyl equation, for many decades neutrinos (= spin 1/2) were thought to be massless
- Outlook:
  - Group-theoretically inspired derivation of Dirac equation:
    spinor representation of Lorentz equation

- Invariance under discrete transformations like: charge conjugation (C), parity (P), time inversion (T)
  > CPT theorem explicitly proven
  > All masses and lifetimes of matter and antimatter are identical
  (Cern experiment: anti-hydrogen)
- Canonical field quantization
- Dirac propagator: important for perturbative calculations

6.1 Pauli Matrices:

- Stern-Gerlach experiment 1922
  1) silver: $5s^1$
  2) hydrogen: $1s^1$
  valence electron



- Explanation: apart from orbital angular momentum the electron also has an internal angular momentum ($S = 1/2$) $\to$ deflection in an inhomogeneous magnetic field
- Mathematical description due to Wolfgang Pauli: 3 2×2-matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2 important properties:

1) Anticommutators: $[\sigma^k, \sigma^l]_+ = 2\delta_{kl} I, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & +1 \end{pmatrix}$
   $\Rightarrow$ Clifford algebra with $N = 3$

N generators $\xi^1, \dots, \xi^N$ form a Clifford algebra: $[\xi^k, \xi^l]_+ = 2\delta_{kl}$

2) Commutators: $[\sigma^k, \sigma^l]_- = 2i\varepsilon_{klm}\sigma^m$

$\Rightarrow$ Lie algebra with $N=3$

N generators $\xi^1, \dots, \xi^N$ form a Lie algebra: $[\xi^k, \xi^l]_- = i\,\underbrace{C_{klm}}_{\text{structure constants}}\xi^m$

1 + 2 :
$$\sigma^k\sigma^l + \sigma^l\sigma^k = 2\delta_{kl}\cdot I$$
$$\sigma^k\sigma^l - \sigma^l\sigma^k = 2i\varepsilon_{klm}\sigma^m$$
$$\sigma^k\sigma^l = \delta_{kl} I + i\varepsilon_{klm}\sigma^m \quad (*)$$

(*) is an important calculational tool to reduce the power of Pauli matrices.

6.2 Spinor representation of Lorentz Algebra: 2 representations!

- Generators of rotations: $L_k = \frac{1}{2}\sigma^k$

$$[L_k, L_l]_- = \frac{1}{4}[\sigma^k, \sigma^l]_- = i\varepsilon_{klm}\left(\frac{1}{2}\sigma^m\right) = i\varepsilon_{klm}L_m \checkmark$$

- Generators of boosts: $M_k = \pm\frac{i}{2}\sigma^k$

1) $[L_k, M_l]_- = \frac{\pm i}{4}[\sigma^k, \sigma^l]_- = i\varepsilon_{klm}\left(\frac{\pm i}{2}\sigma^m\right) = i\varepsilon_{klm}M_m \checkmark$

2) $[M_k, M_l]_- = -\frac{1}{4}[\sigma^k, \sigma^l]_- = -i\varepsilon_{klm}\left(\frac{1}{2}\sigma^m\right) = -i\varepsilon_{klm}L_m \checkmark$

- Two representations:

$D(1/2, 0)$: $(L_k, M_k) = \left(\frac{1}{2}\sigma^k, -\frac{i}{2}\sigma^k\right)$

$D^{(0,1/2)}$: $(L^k, M^k) = \left(\frac{1}{2}\sigma^k, +\frac{i}{2}\sigma^k\right)$

$D^{(s_1,s_2)}$: general representation of Lorentz algebra with $s_1, s_2 = 0, 1/2, 1, 3/2, 2, \ldots$

contains particles with spin $[\,|s_1 - s_2|, \ldots, s_1 + s_2\,]$

$D^{(s,0)}$ or $D^{(0,s)}$: describes particles with fixed spin s

$D^{(0,0)} = 1$

rotation angles rapidities

Lie theorem: $\underbrace{D(\Lambda)}_{\text{representation of Lorentz group}} = e^{-i\vec{L}\cdot\vec{\varphi} - i\vec{M}\cdot\vec{\zeta}}$

representation of Lorentz algebra

$D^{(\frac{1}{2},0)}(\Lambda) = \exp\left\{-\frac{i}{2}\vec{\sigma}\cdot\vec{\varphi} - \frac{1}{2}\vec{\sigma}\vec{\zeta}\right\}$ $\qquad \vec{\varphi} = \vec{0}$: boost

$D^{(0,\frac{1}{2})}(\Lambda) = \exp\left\{-\frac{i}{2}\vec{\sigma}\cdot\vec{\varphi} + \frac{1}{2}\vec{\sigma}\vec{\zeta}\right\}$ $\qquad \vec{\zeta} = 0$: rotation

6.3 Spinor representation of rotation:

$D^{(\frac{1}{2},0)}(R) = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\varphi}} = D^{(0,\frac{1}{2})}(R)$; $(\sigma^k)^+ = \sigma^k$, $\underline{D^+(R) = D^{-1}(R)\text{ unitary}}$

$$D(R(\varphi)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\frac{(\vec{\sigma}\cdot\vec{\varphi})^{2n}}{2^{2n}} - i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{(\vec{\sigma}\cdot\vec{\varphi})^{2n+1}}{2^{2n+1}}$$

$$(\vec{\sigma}\cdot\vec{\varphi})^2 = \varphi_k\varphi_l\,\sigma^k\sigma^l \overset{(*)}{=} \underbrace{\varphi_k\varphi_l}_{\text{symmetric in } k,l}\left(\delta_{kl}\,I + \cancel{i\,\underbrace{\varepsilon_{klm}}_{\text{antisymmetric in } k,l}}\,\sigma^m\right) = \vec{\varphi}^2\cdot I$$

$$D(R(\varphi)) = \left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{|\vec{\varphi}|}{2}\right)^{2n}\right\}\cdot I - i\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{|\vec{\varphi}|}{2}\right)^{2n+1}\right\}\frac{\vec{\sigma}\cdot\vec{\varphi}}{|\vec{\varphi}|}$$

$$\overbrace{\qquad\qquad}^{\cos\left(\frac{|\vec\varphi|}{2}\right)} \qquad\qquad \overbrace{\qquad\qquad}^{\sin\left(\frac{|\vec\varphi|}{2}\right)}$$

$$\Rightarrow \quad D(R(\varphi)) = I \cos\left(\frac{|\vec\varphi|}{2}\right) - i\,\frac{\vec\sigma\cdot\vec\varphi}{|\vec\varphi|}\,\sin\left(\frac{|\vec\varphi|}{2}\right)$$

- unitary as claimed
- rotation of 4π in real space around a fixed axis $\frac{\vec\varphi}{|\vec\varphi|}$ to recover unity

  $\Rightarrow$ characteristic property of a spinor

6.4 Spinor Representation of Boosts:

To evaluate: $D(B(\vec\xi)) = e^{\underset{D^{(0,1/2)}}{\overset{D^{(1/2,0)}}{\mp}} \frac{1}{2}\vec\sigma\cdot\vec\xi} \xrightarrow{\sigma^{+k} = \sigma^k} D^+(B(\vec\xi)) = D(B(\vec\xi))$ hermitian

$$D(B(\vec\xi)) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\,\frac{(\vec\sigma\cdot\vec\xi)^{2n}}{2^{2n}} \mp \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,\frac{(\vec\sigma\cdot\vec\xi)^{2n+1}}{2^{2n+1}}$$

$$\overset{(*)}{=} \underbrace{\left\{\sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{|\vec\xi|}{2}\right)^{2n}\right\}}_{=\cosh\left(\frac{|\vec\xi|}{2}\right)} I \mp \underbrace{\left\{\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{|\vec\xi|}{2}\right)^{2n+1}\right\}}_{\sinh\left(\frac{|\vec\xi|}{2}\right)} \frac{\vec\sigma\cdot\vec\xi}{2}$$

$$\Rightarrow \boxed{D(B(\vec\xi)) = I\cosh\left(\frac{|\vec\xi|}{\cancel{2}}\right) \mp \frac{\vec\sigma\cdot\vec\xi}{|\vec\xi|}\sinh\left(\frac{|\vec\xi|}{\cancel{2}}\right) = e^{\mp\frac{1}{\cancel{2}}\vec\sigma\cdot\vec\xi}} \quad (**)$$

To do: relate rapidity $\vec\xi$ to momentum $\vec p$

$$\underbrace{(p^\mu_R)}_{\text{rest frame}} = \begin{pmatrix} mc \\ \vec 0 \end{pmatrix} \longrightarrow \underbrace{p^\mu = B^\mu{}_\nu(\vec\xi)\, p^\mu_R}_{\text{inertial frame}}$$

$$\left(B^\mu{}_\nu(\vec\xi)\right) = \begin{pmatrix} \boxed{\begin{matrix}\cosh(|\vec\xi|) \\ \frac{\vec\xi}{|\vec\xi|}\sinh(|\vec\xi|)\end{matrix}} & \begin{matrix}\frac{\vec\xi^T}{|\vec\xi|}\sinh|\vec\xi| \\ \delta_{ij} + \frac{\xi^i\xi^j}{|\vec\xi|^2}(\cosh|\vec\xi|-1)\end{matrix}\end{pmatrix} \quad (***)$$

$$\Rightarrow (p^\mu) = \begin{pmatrix} mc\cosh|\vec\xi| \\ mc\frac{\vec\xi}{|\vec\xi|}\sinh|\vec\xi|\end{pmatrix} = \begin{pmatrix} p^0 \\ \vec p\end{pmatrix} = \begin{pmatrix} E/c \\ \vec p\end{pmatrix} \Rightarrow \boxed{\begin{matrix}\cosh|\vec\xi| = \frac{p^0}{mc} \\ \frac{\vec\xi}{|\vec\xi|} = \frac{\vec p}{mc}\frac{1}{\sinh|\vec\xi|}\end{matrix}}$$

$$\cosh\left(\frac{|\vec\xi|}{2}\right) = \sqrt{\frac{\cosh|\vec\xi|+1}{2}} = \sqrt{\frac{p^0+mc}{2mc}} \qquad \sinh\left(\frac{|\vec\xi|}{2}\right) = \sqrt{\frac{\cosh|\vec\xi|-1}{2}} = \sqrt{\frac{p^0-mc}{2mc}}$$

$$\sinh|\vec\xi| = 2\sinh\frac{|\vec\xi|}{2}\cosh\frac{|\vec\xi|}{2} = \frac{\sqrt{(p^0+mc)(p^0-mc)}}{mc}$$

$$D(B(\vec\xi)) = I\sqrt{\frac{p^0+mc}{2mc}} \mp \frac{\vec\sigma\cdot\vec p}{\cancel{mc}}\frac{\cancel{mc}}{\sqrt{(p^0+mc)\cancel{(p^0-mc)}}}\sqrt{\frac{\cancel{p^0-mc}}{2mc}} = \frac{(p^0+mc)I \cancel{-}+ \vec\sigma\cdot\vec p}{\sqrt{2mc(p^0+mc)}}$$

$=\sigma^0$

Four-vector of Pauli matrices:

$$(\sigma^\mu) = (\sigma^0, \sigma^k), \quad \sigma^0 = I = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}; \quad p\cdot\sigma = p_\mu\sigma^\mu = g_{\mu\nu}p^\mu\sigma^\nu = p^0\sigma^0 - \vec p\,\vec\sigma$$

$$D^{(1/2,0)}(B(\vec\xi)) = e^{-\frac{1}{2}\vec\sigma\vec\xi} = \frac{p\cdot\sigma + mc}{\sqrt{2mc(p^0+mc)}}$$

Spatially inverted four vector:

$$(\tilde x) = (\tilde x^0, \tilde x^k) = (x^0, -x^k); \quad \tilde\sigma = (\tilde\sigma^0, \tilde\sigma^k) = (\sigma^0, -\vec\sigma)$$

$$p\cdot\tilde\sigma = g_{\mu\nu}p^\mu\tilde\sigma^\nu = p^0\sigma^0 + \vec p\cdot\vec\sigma$$

$$D^{(0,1/2)}(B(\vec\xi)) = e^{+\frac{1}{2}\vec\sigma\cdot\vec\xi} = \frac{p\cdot\tilde\sigma + mc}{\sqrt{2mc(p^0+mc)}}$$

Note: additional calculation, which turns out to be useful later on

$$e^{\mp\frac{1}{2}\vec{\sigma}\cdot\vec{\zeta}} = \sqrt{e^{\mp\vec{\sigma}\cdot\vec{\zeta}}} \overset{(**)}{=} \sqrt{I\cosh|\vec{\zeta}| \mp \frac{\vec{\sigma}\cdot\vec{\zeta}}{|\vec{\zeta}|}\sinh|\vec{\zeta}|}$$

$$\overset{(***)}{=} \sqrt{\frac{p^0}{mc}\cancel{I} \mp \frac{\vec{\sigma}\cdot\vec{p}}{mc}}$$

$$\underline{e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{\zeta}} = \sqrt{\frac{p^0\sigma^0 - \vec{\sigma}\cdot\vec{p}}{mc}} = \sqrt{\frac{p\sigma}{mc}}}\,, \quad \underline{e^{+\frac{1}{2}\vec{\sigma}\cdot\vec{\zeta}} = \sqrt{\frac{p^0\sigma^0 + \vec{\sigma}\cdot\vec{p}}{mc}} = \sqrt{\frac{p\cdot\tilde{\sigma}}{mc}}}$$

6.5 Lorentz Invariant Combination of Weyl Spinors:

$D^{(1/2,0)}$, $D^{(0,1/2)}$: smallest non-trivial representations of Lorentz group

Weyl spinors:

$$\xi_\alpha(x) \xrightarrow{\Lambda} \xi'_\alpha(x') = D^{(1/2,0)}(\Lambda)_\alpha{}^\beta\, \xi_\beta(x)$$

$$\bar{\xi}^{\dot\alpha}(x) \longrightarrow \bar{\xi}'^{\dot\alpha}(x') = D^{(0,1/2)}(\Lambda)^{\dot\alpha}{}_{\dot\beta}\, \bar{\xi}^{\dot\beta}(x)$$

Aim: search for Lorentz-invariant action based on Weyl spinors

1) quadratic terms in Weyl spinors 2) first derivatives of Weyl spinors

First step: no partial derivative ∃ mass term

$$\xi^+\xi\,,\quad \bar{\xi}^+\bar{\xi}\,,\quad \bar{\xi}^+\xi\,,\quad \xi^+\bar{\xi}$$

$$\xrightarrow{\Lambda} \xi^+ \underbrace{D^{(1/2,0)}(\Lambda)^+}_{D^{-1}(R)} \underbrace{D^{(1/2,0)}(\Lambda)}_{D(R)}\xi\,,\quad \bar{\xi}^+ \underbrace{D^{(0,1/2)+}(\Lambda)}_{D^{-1}(R)} \underbrace{D^{(0,\frac{1}{2})}(\Lambda)}_{D(R)}\bar{\xi}$$

$\Lambda = R$:

$$\xi^+ \underbrace{D^{(0,1/2)^+}(\Lambda)}_{D^{-1}(R)} \underbrace{D^{(\frac{1}{2},0)}(\Lambda)}_{D(R)} \xi, \quad \xi^+ \underbrace{D^{(1/2,0)^+}(\Lambda)}_{D^{-1}(R)} \underbrace{D^{(0,1/2)}(\Lambda)}_{D(R)} \Xi$$

Rotations: $\Lambda = R \Rightarrow D^{(1/2,0)}(R) = D^{(0,1/2)}(R) = D(R), \; D^+(R) = D^{-1}(R)$

All four combinations of Weyl spinors are invariant with respect to rotation

Outlook: $\Lambda = B \Rightarrow$ only $\xi^+ \xi$, $\xi^+ \Xi$ survive

$\hat{=}$ invariant under a boost