two-component Weyl spinors: $\begin{cases} \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_1 \end{pmatrix} : & \zeta' = D^{(1/2,0)}(\Lambda)\,\zeta \\ \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} : & \xi' = D^{(0,1/2)}(\Lambda)\,\xi \end{cases}$

representation

Lorentz transformation

1. Mass term:

$$\zeta^+ \zeta \,, \quad \xi^+ \xi \,, \quad \xi^+ \zeta \,, \quad \zeta^+ \xi$$

$\downarrow$ Lorentz transformation

$$\zeta^+ D^{(1/2,0)}(\Lambda)^+ D^{(1/2,0)}(\Lambda)\,\zeta \,, \quad \xi^+ D^{(0,1/2)}(\Lambda)^+ D^{(0,1/2)}(\Lambda)\,\xi$$

$$\underbrace{\xi^+ D^{(0,1/2)}(\Lambda)^+ D^{(1/2,0)}(\Lambda)\,\zeta}_{=\xi^+\zeta} \,, \quad \underline{\zeta^+ D^{(1/2,0)}(\Lambda)^+ D^{(0,1/2)}(\Lambda)\,\xi} = \zeta^+\xi$$

last time: rotation $\Lambda = R$, $D^{(1/2,0)}(R) = D^{(0,1/2)}(R) = D(R)$

and $D^+(R)\,D(R) = \mathbb{1}$

$\Rightarrow$ all 4 terms are invariant with respect to rotations

now: boost $\Lambda = B$, $D^{(1/2,0)}(B) = e^{-\frac{1}{2}\vec{\sigma}\vec{\zeta}}$, $D^{(0,1/2)}(B) = e^{+\frac{1}{2}\vec{\sigma}\vec{\zeta}}$

$\Rightarrow D^{(1/2,0)}(B) = D^{(0,1/2)}{}^{-1}(B)$, $D(B) = D(B)^+$

$$D^{(1/2,0)}(B)^+ D^{(0,1/2)}(B) = D^{(0,1/2)}(B)^{-2} D^{(0,1/2)}(B) = \mathbb{1}$$

$D^{(0,1/2)}(B) + D^{(1/2,0)}(B) \neq 1$

$\Rightarrow$ only $\xi^+\xi$, $\zeta^+\zeta$ term are invariant under boosts

2) Terms with first partial derivatives:

$\xi^+(\underline{\sigma^k \partial_k})\xi$, $\zeta^+(\underline{\sigma^k\partial_k})\zeta$, $\zeta^+(\underline{\sigma^k\partial_k})\xi$, $\xi^+(\underline{\sigma^k\partial_k})\zeta$

$\xi^+ D^{(1/2,0)}(\Lambda)^+ \sigma^k D^{(1/2,0)}(\Lambda)\partial_k \xi$, $\zeta^+ D^{(0,1/2)}(\Lambda)\sigma^k D^{(0,1/2)}(\Lambda)\partial_k\zeta$, ...

rotations $\Lambda = R \Rightarrow \boxed{D(R)^+ \sigma^k D(R) = ?}$

$= \left\{\cos\left(\frac{|\vec\varphi|}{2}\right) + i\frac{\vec\sigma\vec\varphi}{|\vec\varphi|}\sin\left(\frac{|\vec\varphi|}{2}\right)\right\}\sigma^k\left\{\cos\left(\frac{|\vec\varphi|}{2}\right) - i\frac{\vec\sigma\vec\varphi}{|\vec\varphi|}\sin\left(\frac{|\vec\varphi|}{2}\right)\right\}$

$= \underbrace{\cos^2\left(\frac{|\vec\varphi|}{2}\right)}_{=\frac{1}{2}(1+\cos|\vec\varphi|)}\sigma^k + i\underbrace{\sin\left(\frac{|\vec\varphi|}{2}\right)\cos\left(\frac{|\vec\varphi|}{2}\right)}_{=\frac{1}{2}\sin|\vec\varphi|}\frac{\varphi_l}{|\vec\varphi|}\underbrace{[\sigma^l,\sigma^k]_-}_{=2i\varepsilon_{lkm}\sigma^m} + \underbrace{\sin^2\left(\frac{|\vec\varphi|}{2}\right)}_{=\frac{1}{2}(1-\cos|\vec\varphi|)}\frac{\varphi_l\varphi_m}{|\vec\varphi|^2}\underbrace{(\sigma^l\sigma^k)\sigma^m}_{=?}$

$\longrightarrow = (\delta_{lk} + i\varepsilon_{lkn}\sigma^n)\sigma^m = \delta_{lk}\sigma^m + i\varepsilon_{lkn}\underbrace{\sigma^n\sigma^m}_{=\delta_{nm}+i\varepsilon_{nmp}\sigma^p}\sigma^m$

$= \delta_{lk}\sigma^m + i\varepsilon_{lkn}\delta_{nm}\sigma^m - \underbrace{\varepsilon_{lkn}\varepsilon_{nmp}}_{=\varepsilon_{lkn}\varepsilon_{mpn} = \delta_{lm}\delta_{kp} - \delta_{lp}\delta_{km}}\sigma^p$

$= \ldots = \delta_{lk}\sigma^m + i\varepsilon_{lkm}\sigma^m - \delta_{lm}\sigma^k + \delta_{km}\sigma^l$

$$\Rightarrow D(R)^{+} \sigma^{k} D(R) = \cos|\vec{\varphi}|\, \sigma^{k} + \varepsilon_{klm} \frac{\varphi_l}{|\vec{\varphi}|} \sigma^{m} + (1-\cos|\vec{\varphi}|) \frac{\varphi_k \vec{\sigma}\vec{\varphi}}{|\vec{\varphi}|^2}$$

$$= \underbrace{R_{kl}}_{\text{representation matrix for rotation in 3D, see exercises}} \sigma^{l}$$

$$D(R)^{+} \sigma^{k} D(R) \partial_k = R_{kl} \sigma^{l} R_{km} \partial_m \underset{\text{orthonormality}}{=} \delta_{lm} \sigma^{l} \partial_m = \sigma^{l} \partial_l$$

orthonormality $R_{kl} R_{km} = (R^T R)_{lm} = \delta_{lm}$

$\Rightarrow$ also here all 4 terms are invariant with respect to rotations

boost: $\sigma^k \longrightarrow$ four vector $\nearrow (\sigma^\mu) = (\sigma^0, \sigma^k)$

$\searrow (\tilde{\sigma}^\mu) = (\sigma^0, -\sigma^k)$

eight combinations to consider

invariant!

excluded

$\cancel{\xi^+ \sigma^\mu \partial_\mu \xi}$, $\xi^+ \tilde{\sigma}^\mu \partial_\mu \xi$, $\eta^+ \sigma^\mu \partial_\mu \eta$, $\cancel{\eta^+ \tilde{\sigma}^\mu \partial_\mu \eta}$,

$\cancel{\xi^+ \sigma^\mu \partial_\mu \eta}$, $\cancel{\xi^+ \tilde{\sigma}^\mu \partial_\mu \eta}$, $\cancel{\eta^+ \sigma^\mu \partial_\mu \xi}$, $\cancel{\eta^+ \tilde{\sigma}^\mu \partial_\mu \xi}$

additional terms with $\sigma^0 \partial_0$ should not destroy invariance under rotations:

$$D(R)^{+} \underbrace{\sigma^0}_{=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \partial_0 D(R) = \underbrace{D(R)^{+}}_{\overset{\text{unitarity}}{=} D(R)^{-1}} D(R) \partial_0 = \partial_0 \quad \checkmark$$

now: boosts?

$$D(B)^+ \sigma^\mu D(B) \qquad \text{and} \qquad D(B)^+ \tilde\sigma^\mu D(B)$$

$D^{(0,1/2)}(B)$ or $D^{(1/2,0)}(B)$

$\mu = 0$: different boost representations left and right

$$D^{(1/2,0)+}(B)\, \sigma^0\, D^{(0,1/2)}(B) = \underbrace{D^{(1/2,0)+}(B)}_{D^{(0,1/2)}(B)^{-1}} D^{(0,1/2)}(B) = \sigma^0$$

$$D^{(0,1/2)+}(B)\, \sigma^0\, D^{(1/2,0)}(B) = \;---\; = \sigma^0$$

This cannot be!

now: left and right boost representation are the same

$\mu = 0$: $\underbrace{D(B)^+}_{\substack{= D(B) \\ \uparrow \\ \text{hermiticity}}} \underbrace{\sigma^0}_{\text{identity}} D(B) = D(B)^2 = e^{\mp \frac{1}{2} \vec\sigma \vec\xi \cdot 2} = e^{\mp \vec\sigma \vec\xi}$

$D^{(1/2,0)}(B)$ $\quad$ $D^{(0,1/2)}(B)$

$$= \cosh|\vec\xi| \mp \frac{\vec\sigma \vec\xi}{|\vec\xi|} \sinh|\vec\xi|$$

last lecture

$\mu = k$: $\underbrace{D(B)^+}_{= D(B)} \sigma^k D(B) = \left\{\cosh\frac{|\vec\xi|}{2} \mp \frac{\vec\sigma\vec\xi}{|\vec\xi|} \sinh\frac{|\vec\xi|}{2}\right\} \sigma^k \left\{\cosh\left(\frac{|\vec\xi|}{2}\right) \mp \frac{\vec\sigma\vec\xi}{|\vec\xi|} \sinh|\vec\xi|\right\}$

$$= \cosh^2\left(\frac{|\vec\xi|}{2}\right)\sigma^k \mp \sinh\frac{|\vec\xi|}{2}\cosh\frac{|\vec\xi|}{2}\frac{\xi_l}{|\vec\xi|}[\sigma^l,\sigma^k]_+ + \sinh^2\frac{|\vec\xi|}{2}\frac{\xi_l\xi_m}{|\vec\xi|^2}\sigma^l\sigma^k\sigma^m$$

$$\tfrac{1}{2}(\cancel{1}\cosh|\vec\xi|) \qquad = \tfrac{1}{2}\sinh|\vec\xi| \qquad 2\,\delta_{k\ell} \qquad \tfrac{1}{2}(\cosh|\vec\xi|-1) \qquad \text{see above}$$

$$= \dots = \sigma^k \mp \frac{\xi_k}{|\vec\xi|}\sinh|\vec\xi| + \frac{\xi_k}{|\vec\xi|}\frac{\vec\xi\vec\sigma}{|\vec\xi|}(\cosh|\vec\xi|-1)$$

$$D(B)^+(\mp\sigma^k)D(B) = (\mp\sigma^k) + \frac{\xi_k}{|\vec\xi|}\sinh|\vec\xi| + (\cosh|\vec\xi|-1)\frac{\xi_k}{|\vec\xi|}\frac{\vec\xi\cdot(\mp\vec\sigma)}{|\vec\xi|} \Leftarrow$$

$$(B^\mu{}_\nu) = \begin{pmatrix} \cosh|\vec\xi| & \frac{\xi_k}{|\vec\xi|}\sinh|\vec\xi| \\ \frac{\xi_k}{|\vec\xi|}\sinh|\vec\xi| & \delta_{k\ell} + (\cosh|\vec\xi|-1)\frac{\xi_k\xi_\ell}{|\vec\xi|^2} \end{pmatrix} \qquad \begin{pmatrix} \sigma^0 \\ \mp\sigma^k \end{pmatrix}$$

Result:

1) $D^{(\frac{1}{2},0)}(B)^+\tilde\sigma^\mu D^{(\frac{1}{2},0)}(B) = B^\mu{}_\nu\,\tilde\sigma^\nu$

2) $D^{(0,\frac{1}{2})}(B)^+\sigma^\mu D^{(0,1/2)}(B) = B^\mu{}_\nu\,\sigma^\nu$

case 2): $\xi^+\sigma^\mu\partial_\mu\xi \xrightarrow{B} \left(D^{(0,1/2)}(B)^+\sigma^\mu D^{(0,1/2)}(B)\right)\partial'_\mu$

$$\overset{2)}{=} B^\mu{}_\nu\sigma^\nu B_\mu{}^\varkappa\partial_\varkappa = \underbrace{B^\mu{}_\nu B_\mu{}^\varkappa}_{=?\;\delta_\nu{}^\varkappa}\sigma^\nu\partial_\varkappa = \sigma^\varkappa\partial_\varkappa \checkmark$$

definition of Lorentz transformation:

$$\Lambda^T g\Lambda = g\,, \quad \Lambda^\sigma{}_\mu g_{\sigma\rho}\Lambda^\rho{}_\nu = g_{\mu\nu} \quad \Big|\; g^{\nu\varkappa}$$

$$g_{\mu\nu}g^{\nu\varkappa} = g_\mu{}^\varkappa = \delta_\mu{}^\varkappa = \Lambda^\sigma{}_\mu g_{\sigma\rho}\Lambda^\rho{}_\nu g^{\nu\varkappa}$$

$$\overbrace{\Lambda^{\rho\kappa}}$$

$$= \underbrace{\partial_\sigma^{\ \rho}}_{=\delta_\sigma^{\ \rho}} \Lambda_\rho^{\ \kappa} = \Lambda_\sigma^{\ \kappa}$$

$$\Rightarrow \delta_\mu^{\ \kappa} = \Lambda^\sigma_{\ \mu} \Lambda_\sigma^{\ \kappa}$$

$$E_{4\times 4} = \Lambda^T \Lambda \quad \Rightarrow \quad \boxed{\Lambda^T = \Lambda^{-1}}$$

$$e^{-i\vec{\lambda}\vec{\varphi}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} \quad \longrightarrow \quad R^T = R^{-1}$$

6.6 Dirac Action:

$$S = \frac{1}{c}\int d^4x\, \mathcal{L}\left(\xi(x), \partial_\mu \xi(x); \xi^+(x), \partial_\mu \xi^+(x); \zeta(x), \partial_\mu \zeta(x); \zeta^+(x), \partial_\mu \zeta^+(x)\right)$$

$$\Rightarrow \mathcal{L} = A\, i\, \xi^+ \tilde{\sigma}^\mu \partial_\mu \xi + B\, i\, \zeta^+ \sigma^\mu \partial_\mu \zeta + C\, \xi^+ \zeta + D\, \zeta^+ \xi^+$$

A, B, C, D: not determined from group theory

later (Section 6.8): parity transformation

$\Rightarrow$ ξ and ζ have to appear in the same way

$$\mathcal{L} = A\left\{ \underline{i\, \xi^+ \tilde{\sigma}^\mu \partial_\mu \xi} + \underline{i\, \zeta^+ \sigma^\mu \partial_\mu \zeta} - m\, \xi^+ \zeta - m\, \zeta^+ \xi \right\}$$

kinetic terms — mass terms

Note: massive spin 1/2: $m \neq 0 \Rightarrow$ both ξ and χ are present

massless spin 1/2: $m = 0 \Rightarrow$ only one Weyl spinor is possible provided that parity breaks

Equations of motion:

$$\frac{\delta A}{\delta \xi^+(x)} = 0 = \frac{\partial \mathcal{L}}{\partial \xi^+(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \xi^+(x))} = 0 \Rightarrow i\tilde{\sigma}^\mu \partial_\mu \xi - m\chi = 0 \quad (1)$$

$$\frac{\delta A}{\delta \chi^+(x)} = 0 \qquad \Rightarrow \quad i\sigma^\mu \partial_\mu \chi - m\xi = 0 \quad (2)$$

Calculational tool:

$$\sigma^\mu \tilde{\sigma}^\nu + \sigma^\nu \tilde{\sigma}^\mu = 2g^{\mu\nu} I \quad (*) \quad \tilde{\sigma}^\mu \sigma^\nu + \tilde{\sigma}^\nu \sigma^\mu = 2g^{\mu\nu} I$$

follows from Clifford algebra of Pauli matrices and Minkowski metric:

$$\tilde{\sigma}^0 \sigma^0 + \sigma^0 \tilde{\sigma}^0 = \sigma^0\sigma^0 + \sigma^0\sigma^0 = 2g^{00} I \quad \checkmark$$

$$\sigma^0 \tilde{\sigma}^k + \sigma^k \tilde{\sigma}^0 = -\sigma^0\sigma^k + \sigma^k\sigma^0 = 0 = 2g^{0k} I \quad \checkmark$$

$$\sigma^k \tilde{\sigma}^l + \sigma^l \tilde{\sigma}^k = -[\sigma^k, \sigma^l]_+ = -2\delta_{kl} I = 2g^{kl} I \quad \checkmark$$

$$(1) \quad \Big| \; i\sigma^\nu \partial_\nu$$

$$\underbrace{-\sigma^\nu \tilde{\sigma}^\mu \underset{\text{sym. }\nu,\mu}{\partial_\nu\partial_\mu}}_{\frac{1}{2}(\sigma^\nu\tilde{\sigma}^\mu + \sigma^\mu\tilde{\sigma}^\nu) \overset{(\times)}{=} g^{\mu\nu}\,\mathrm{I}} \zeta \; - m\, \underbrace{i\sigma^\nu \partial_\nu \xi}_{\overset{(2)}{=} m\zeta} = 0 \quad \Rightarrow (\underbrace{g^{\mu\nu}\partial_\mu\partial_\nu}_{=\Box} - m^2)\zeta = 0$$

Klein-Gordon equation

Identification: $m = \frac{Mc}{\hbar}$ $\hat{=}$ inverse Compton wave length

Suggestive to combine both Weyl-spinors into Dirac spinor

$$\Psi(x) = \begin{pmatrix} \zeta(x) \\ \xi(x) \end{pmatrix} \qquad \mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{L} = A\Big\{ \underbrace{(\zeta^+, \xi^+)}_{=\Psi^+} \left(\begin{array}{c|c} \tilde{\sigma}^\mu & \mathbf{O} \\ \hline \mathbf{O} & \sigma^\mu \end{array}\right) i\partial_\mu \begin{pmatrix} \zeta \\ \xi \end{pmatrix} - (\zeta^+, \xi^+) \left(\begin{array}{c|c} \mathbf{O} & m\mathrm{I} \\ \hline m\mathrm{I} & \mathbf{O} \end{array}\right) \begin{pmatrix} \zeta \\ \xi \end{pmatrix} \Big\}$$

Dirac adjoint Dirac spinor

$$\overline{\Psi}(x) = \underbrace{\Psi^+(x)}_{(\zeta^+, \xi^+)} \begin{pmatrix} \mathbf{O} & \mathrm{I} \\ \mathrm{I} & \mathbf{O} \end{pmatrix} = (\xi^+(x), \zeta^+(x))$$

$$\Psi^+(x) = \overline{\Psi}(x) \begin{pmatrix} \mathbf{O} & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix}$$

$$\mathcal{L} = A\Big\{ \overline{\Psi} \underbrace{\begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}}_{= \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} = \gamma^\mu} i\partial_\mu \Psi - \overline{\Psi} \underbrace{\begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix} \begin{pmatrix} 0 & m\mathrm{I} \\ m\mathrm{I} & 0 \end{pmatrix}}_{= m\begin{pmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{pmatrix} \,\hat{=}\, m} \Psi \Big\}$$

dirac matrix

short-hand notation

$$\Rightarrow \quad \mathcal{L} = A \quad \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$$

clifford algebra

$$[\gamma^\mu, \gamma^\nu]_+ = \dots = 2g^{\mu\nu}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

short-hand notation

$$= 2g^{\mu\nu}$$

$$\frac{\delta A}{\delta \bar{\Psi}(x)} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \bar{\Psi}(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi}(x))} = 0 \Rightarrow (i\underbrace{\gamma^\mu \partial_\mu}_{\not{\partial}} - m)\Psi = 0$$

Feynmandagger