6.7 Spinor Representation of Dirac Theory:

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix} \xrightarrow{\Lambda} \psi'(x') = \begin{pmatrix} \xi'(x') \\ \eta'(x') \end{pmatrix} = \underbrace{D(\Lambda)}\, \psi(x)$$

representation matrix of Lorentz transformation Λ in space of Dirac spinors ψ

$$D(\Lambda) = \left(\begin{array}{c|c} D^{(1/2,0)}(\Lambda) & 0 \\ \hline 0 & D^{(0,1/2)}(\Lambda) \end{array}\right)$$

Dirac adjoint Dirac spinor: $\overline{\psi}(x) = (\eta^+(x), \xi^+(x)) = \psi^+(x) \underbrace{\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}_{=\gamma^0}$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \Rightarrow \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\overline{\psi}(x) = \psi^+(x)\gamma^0 \Longleftrightarrow \psi^+(x) = \overline{\psi}(x)\gamma^0, \quad (\gamma^0)^2 = 1$$

$$\psi'^+(x') = \psi^+(x) D^+(\Lambda)$$

$$\overline{\psi}'(x') = \psi^{+\prime}(x')\gamma^0 = \psi^+(x) D^+(\Lambda)\gamma^0 = \overline{\psi}(x) \underbrace{\gamma^0 D^+(\Lambda)\gamma^0}_{=\overline{D}(\Lambda)}$$

$$\bar D(\Lambda) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} D^{(1/2,0)+}(\Lambda) & 0 \\ 0 & D^{(0,1/2)+}(\Lambda) \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} D^{(0,1/2)+}(\Lambda) & 0 \\ 0 & D^{(1/2,0)+}(\Lambda) \end{pmatrix}$$

$$D^{(1/2,0)}(\Lambda) = \exp\left\{ -\frac{i}{2}\vec{\sigma}\cdot\vec{\varphi} - \frac{1}{2}\vec{\sigma}\cdot\vec{\xi} \right\} \quad ; \quad D^{(1/2,0)}(\Lambda)^{-1} = D^{(0,1/2)}(\Lambda)^{+}$$

$$D^{(0,1/2)}(\Lambda) = \exp\left\{ -\frac{i}{2}\vec{\sigma}\cdot\vec{\varphi} + \frac{1}{2}\vec{\sigma}\cdot\vec{\xi} \right\} \quad ; \quad D^{(0,1/2)}(\Lambda)^{-1} = D^{(1/2,0)}(\Lambda)^{+}$$

$$\Rightarrow \quad \bar D(\Lambda) = D^{-1}(\Lambda)$$

section 6.5 for $\Lambda = R$ or $\Lambda = B$

$$D^{(1/2,0)}(\Lambda)\,\tilde\sigma^\mu\, D^{(1/2,0)}(\Lambda) = \Lambda^\mu{}_\nu\, \tilde\sigma^\nu$$

$$D^{(0,1/2)}(\Lambda)\,\sigma^\mu\, D^{(0,1/2)}(\Lambda) = \Lambda^\mu{}_\nu\, \sigma^\nu$$

are also valid for any Λ

decomposition: $\Lambda = B\,R$

$$D^{(1/2,0)}_{(0,1/2)}(\Lambda) = D^{(1/2,0)}_{(0,1/2)}(B\,R) = D^{(1/2,0)}_{(0,1/2)}(B)\, D^{(1/2,0)}_{(0,1/2)}(R)$$

$$D^{(1/2,0)}(\Lambda)\,\tilde\sigma^\mu\, D^{(1/2,0)}(\Lambda) = D^{(1/2,0)}(B)\,\underbrace{D^{(1/2,0)}(R)\,\tilde\sigma^\mu\, D^{(1/2,0)}(R)}_{R^\mu{}_\nu\,\tilde\sigma^\mu}\, D^{(1/2,0)}(B)$$

$$= \left(R^\mu{}_\nu\, B^\nu{}_\kappa\right)\tilde\sigma^\kappa = \Lambda^\mu{}_\kappa\, \tilde\sigma^\kappa$$

the same for σ^μ

$$\bar D(\Lambda)\,\gamma^\mu\, D(\Lambda) = \begin{pmatrix} D^{(0,1/2)+}(\Lambda) & 0 \\ 0 & D^{(1/2,0)+}(\Lambda) \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \tilde\sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} D^{(1/2,0)}(\Lambda) & 0 \\ 0 & D^{(0,1/2)}(\Lambda) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \underline{D^{(0,1/2)\dagger}(\Lambda)\, \sigma^\mu\, D^{(0,1/2)}(\Lambda)} \\ \underline{D^{(1/2,0)\dagger}(\Lambda)\, \tilde{\sigma}^\mu\, D^{(1/2,0)}(\Lambda)} & 0 \end{pmatrix}$$

$$= \Lambda^\mu{}_\nu \begin{pmatrix} 0 & \sigma^\nu \\ \tilde{\sigma}^\nu & 0 \end{pmatrix} = \Lambda^\mu{}_\nu\, \gamma^\nu$$

Dirac matrices transform like a vector

Invariance of Dirac action under Lorentz transformations

$$A = \frac{\hbar}{c} \int d^4x\, \bar{\Psi}(x) \left(i \gamma^\mu \partial_\mu - m \right) \Psi(x)$$

$$A' = \frac{\hbar}{c} \int \underbrace{d^4x'}_{= d^4x} \bar{\Psi}'(x') \left(i \gamma^\mu \partial'_\mu - m \right) \Psi'(x')$$

$$\bar{\Psi}(x) \Big\{ i \underbrace{\underbrace{\bar{D}(\Lambda)\gamma^\mu D(\Lambda)}_{\Lambda^\mu{}_\nu \gamma^\nu}\, \underbrace{\partial'_\mu}_{\Lambda_\mu{}^\lambda \partial_\lambda}}_{\Lambda^\mu{}_\nu \Lambda_\mu{}^\lambda\, \gamma^\nu \partial_\lambda} - m \underbrace{\underbrace{\bar{D}(\Lambda)}_{D^{-1}(\Lambda)} D(\Lambda)}_{= 1} \Big\} \Psi(x)$$

special Lorentz transformation

$$\gamma^\nu \partial_\nu = \underbrace{\Lambda^\mu{}_\nu \Lambda_\mu{}^\lambda}_{= \delta_\nu{}^\lambda} \gamma^\nu \partial_\lambda$$

$$\Rightarrow A' = A$$

$$D(\Lambda) = \begin{pmatrix} e^{-\frac{i}{2}\vec{\sigma}\vec{\varphi} \overset{(-i)(-i)}{-} \underline{\frac{1}{2}\vec{\sigma}\vec{\xi}}} & 0 \\ 0 & e^{-\frac{i}{2}\vec{\sigma}\vec{\varphi} \overset{=(-i)(+i)}{+} \underline{\frac{1}{2}\vec{\sigma}\vec{\xi}}} \end{pmatrix} = e^{-\frac{i}{2} \underbrace{\omega_{\mu\nu}}_{\text{angles, rapidities}} S^{\mu\nu}} \quad (*)$$

representation of spin angular momentum

$$D(\Lambda) = \exp\left\{ -\frac{\varepsilon}{2}\,2\left[\underbrace{\omega_{0i}}_{=\xi_i} S^{0i} + \cancel{\underbrace{\omega_{i0}}_{=-\omega_{0i}} \underbrace{S^{i0}}_{=-S^{0i}}} \right] - \frac{\varepsilon}{2} \underbrace{\omega_{ij}}_{=\varepsilon_{ijk}\varphi^k} S^{ij} \right\}$$

$$D(M_k) = S^{0k} = \begin{pmatrix} -i\sigma^k/2 & 0 \\ 0 & +i\sigma^k/2 \end{pmatrix}$$

$$D(L_k) = \underbrace{S^k} = \frac{1}{2}\varepsilon_{kij} S^{ij} = \boxed{\begin{pmatrix} \sigma^k/2 & 0 \\ 0 & \sigma^k/2 \end{pmatrix}}$$

spin vector of spin 1/2
particle, see Poincaré chapter

Covariant consequence: $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]_-$

specialize μ, ν : $0, i$ and i, j

Useful formulas:

$[S^{\mu\nu}, \gamma^\lambda]_- = i(g^{\nu\lambda}\gamma^\mu - g^{\mu\lambda}\gamma^\nu) \rightarrow \gamma^\lambda$ tensor of 1st rank

$[S^{\mu\nu}, \underbrace{S^{\kappa\lambda}}]_- = i(g^{\mu\lambda} S^{\nu\kappa} + g^{\nu\kappa} S^{\mu\lambda} - g^{\mu\kappa} S^{\nu\lambda} - g^{\nu\lambda} S^{\mu\kappa})$

$S^{\kappa\lambda} \rightarrow$ tensor of 2nd rank

6.8 Parity Transformation:

$$x = (x^0, x^k) \longrightarrow x'_P = P\,x = \tilde{x} = (x^0, -x^k)$$

involutoric: $\quad P^2 = 1 \quad \Leftrightarrow \quad P^{-1} = P$

representation matrix: $\quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

explicit calculation: $\quad P^{-1} L_k P = \ldots = L_k \quad \Rightarrow \quad [L_k, P]_- = 0$

$$P^{-1} M_k P = \ldots = -M_k \quad \Rightarrow \quad [M_k, P]_+ = 0$$

straight-forward

Parity transformation for Dirac spinors:

$$\psi(x) \longrightarrow \psi'_P(x) = \underbrace{D(P)}\,\psi(\tilde{x})$$

representation matrix of parity transformation in space of Dirac spinors

demand for $D(P)$: $\quad D(P)^2 = 1$

$$D^{-1}(P)\,D(L_k)\,D(P) = D(L_k)$$

$$D^{-1}(P)\,D(M_k)\,D(P) = -D(M_k)$$

determine $D(P)$ from invariance of Dirac equation and parity:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \xrightarrow{P} (i\underbrace{\gamma^\mu \tilde{\partial}_\mu}_{=\tilde{\gamma}^\mu \partial_\mu} - m)\underbrace{\psi(\tilde{x})}_{D^{-1}(P)\psi'_P(x)} = 0 \quad \Big| D(P)\cdot$$

$$\Big\{ i\underbrace{D(P)\tilde{\gamma}^\mu D^{-1}(P)}_{=\gamma^\mu}\partial_\mu - m\underbrace{D(P)D^{-1}(P)}_{=1}\Big\}\psi'_P(x) = 0$$

Condition for $D(P)$!

$\Leftarrow$ Then $\psi'_P(x)$ and $\psi(x)$ fulfill the same Dirac equation.

Claim: $D(P) = \gamma^0$

1) $D(P)^2 = (\gamma^0)^2 = 1$

2) $D(P)\tilde{\gamma}^\mu D^{-1}(P) = \gamma^\mu$

$\mu = 0$: $D(P)\tilde{\gamma}^0 D^{-1}(P) = \gamma^0(\gamma^0\gamma^0) = \gamma^0$ ✓

$\mu = k$: $D(P)\tilde{\gamma}^k D^{-1}(P) = -\gamma^0\gamma^k\gamma^0 = \gamma^k$ ✓

$$\underline{\underline{[\gamma^0, \gamma^k]_+ = 2g^{0k} = 0}}$$

$$\gamma^0\gamma^k + \gamma^k\gamma^0 = 0 \quad |\gamma^0\cdot$$

$$\gamma^k + \gamma^0\gamma^k\gamma^0 = 0$$

3) $D(P)^{-1}D(CR)D(P) = D(CR)$ ✓

4) $D(P)D(MR)D(P) = -D(MR)$ ✓

Effect of $D(P)$: $\psi(x) = \begin{pmatrix} \eta(x) \\ \xi(x) \end{pmatrix} \xrightarrow{P} \psi'_P(x) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \eta(\tilde{x}) \\ \xi(\tilde{x}) \end{pmatrix} = \begin{pmatrix} \xi(\tilde{x}) \\ \eta(\tilde{x}) \end{pmatrix}$

parity transformation exchanges both Weyl spinors

- In a theory where both $\psi(x)$ and $\psi_P(x)$ are theoretically realized
  $\Rightarrow$ both Weyl spinors appear

  used in sect. 6.5 for constructing the mass term

- parity transformation invariant theory $\hat{=}$ ζ and ξ appear on equal footing

6.9 Neutrinos:

- spin 1/2 particles interact only via weak interaction and gravity
- postulated neutrinos in 1930 by Wolfgang Pauli in order to explain that electrons resulting from β-decay have a continuous spectrum and not a fixed energy ($\hat{=}$ energy/momentum conservation)
- neutrinos were considered as massless spin 1/2 particles

  1) $\mathcal{L} = A\, i\, \zeta^+ \underline{\bar{\sigma}^\mu} \underline{\partial_\mu} \zeta$ 2) $\mathcal{L} = A\, i\, \xi^+ \underline{\sigma^\mu} \underline{\partial_\mu} \xi$

  invariant under Lorentz transformations but not under parity transformation

description of Weyl spinors from point of view of Dirac spinors

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \ldots = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

projection matrices:

$$P_u \text{ (upper)} = \frac{1}{2}(1-\gamma^5) = \ldots = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \; ; \; P_u\psi = \begin{pmatrix} \eta \\ 0 \end{pmatrix}$$

$$P_e \text{ (lower)} = \frac{1}{2}(1+\gamma^5) = \ldots = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \; ; \; P_e\psi = \begin{pmatrix} 0 \\ \xi \end{pmatrix}$$

Observation: $\underbrace{\gamma^5}_{\text{chirality operator}} \frac{1}{2}(1\mp\gamma^5) = \mp \frac{1}{2}(1\mp\gamma^5)$

+1 (right) chirality

−1 (left) chirality

Claim: $\gamma^5 = \frac{i}{24} \underbrace{\varepsilon_{\mu\nu\varkappa\lambda}}_{\text{anti-symmetry: 24 terms} \to \text{all identical}} \gamma^\mu\gamma^\nu\gamma^\varkappa\gamma^\lambda = i\gamma^0\gamma^1\gamma^2\gamma^3$

Lorentz invariance:

$$\bar{D}(\Lambda)\gamma^5 D(\Lambda) = \frac{i}{24}\varepsilon_{\mu\nu\varkappa\lambda}\underbrace{[\bar{D}(\Lambda)\gamma^\mu D(\Lambda)]}_{\Lambda^\mu{}_{\mu'}\gamma^{\mu'}}\underbrace{[\bar{D}(\Lambda)\gamma^\nu D(\Lambda)]}_{\Lambda^\nu{}_{\nu'}\gamma^{\nu'}}$$

$$\cdot \underbrace{[\bar{D}(\Lambda)\gamma^\varkappa D(\Lambda)]}_{\Lambda^\varkappa{}_{\varkappa'}\gamma^{\varkappa'}}\underbrace{[\bar{D}(\Lambda)\gamma^\lambda D(\Lambda)]}_{\Lambda^\lambda{}_{\lambda'}\gamma^{\lambda'}}$$

$=1$

Weierstraß expansion for determinant of Λ:

$$\underbrace{(\det\Lambda)}_{=1 \text{ special Lorentz transformation}}\varepsilon_{\mu'\nu'\varkappa'\lambda'} = \varepsilon_{\mu\nu\varkappa\lambda}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\Lambda^\varkappa{}_{\varkappa'}\Lambda^\lambda{}_{\lambda'}$$

$$\overline{D}(\Lambda)\gamma^5 D(\Lambda) = \frac{i}{24}\varepsilon_{\mu'\nu'\kappa'\lambda'}\gamma^{\mu'}\gamma^{\nu'}\gamma^{\kappa'}\gamma^{\lambda'} = \gamma^5$$

Neutrino Lagrangian:

$$\mathcal{L} = \hbar i \; \overline{\Psi}(x)\gamma^\mu \partial_\mu \frac{1}{2}(1 \mp \gamma^5)\Psi(x)$$

$-$ $\hat{=}$ 1) with ξ $\longleftrightarrow$ $+$ $\hat{=}$ 2) with ξ

both neutrino actions are Lorentz invariant

but parity transformation $\longleftrightarrow$

History:

- neutrino actions postulated by Hermann Weyl in 1929 to describe massless spin 1/2 particles
- At that time only parity transformation invariant QFT were known so only " " " theories were considered to be physical
- 1956: Chien-Shiung Wu, β-decay experiment of $^{60}_{27}Co$ weak interaction is not invariant under parity transformations
- since then neutrinos were described by Weyl actions
- 1987 Kamiokande experiment $\rightarrow$ neutrino oscillations between

electron-, myon- tauon-neutrinos were observed

- Conclusion: neutrinos do have a mass
- Due to charge neutrality there are two possible descriptions
  - > Paul Dirac: neutrino and anti-matter neutrino are different particles
  - > Ettore Majorana: neutrino and anti-matter neutrino are one and the same particle

=> Experimental decision between both possible theoretical descriptions is still lacking.