neutrinos without rest mass:

$$\psi = \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \nearrow \frac{1}{2}(1-\gamma 5)\psi = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$$
$$\searrow \frac{1}{2}(1+\gamma 5)\psi = \begin{pmatrix} 0 \\ \zeta \end{pmatrix}$$

$$\boxed{\gamma^5 \, \frac{1}{2}(1\mp\gamma^5)\psi = \underbrace{\mp}_{\text{chirality}} \frac{1}{2}(1\mp\gamma^5)\psi}$$

chirality operator $\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$

action: $A = \int \overline{\psi} \, i\gamma^5 \partial_\mu \, \frac{1}{2}(1\mp\gamma 5)\psi$

no rest mass term mixing ξ and ζ

Weyl equation:

$$\frac{\delta A}{\delta \overline{\psi}(x)} = \frac{\partial \mathcal{L}}{\partial \overline{\psi}(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \overline{\psi}(x)} = 0 \Rightarrow i\gamma^\mu \partial_\mu \frac{1}{2}(1\mp\gamma 5)\psi = 0$$

no $\hbar$ here

fixed momentum: $\psi(x) = \psi \, e^{-ipx/\hbar}$

$$\Rightarrow \vec{\gamma}\cdot\vec{p} \, \frac{1}{2}(1\mp\gamma^5)\psi = \gamma^0 p^0 \, \frac{1}{2}(1\mp\gamma 5)\psi \quad \Big| \; \gamma^5\gamma^0 \cdot$$

1) $\gamma^5 \gamma^0 \gamma^k = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$

2) $\gamma^5(\gamma^0\gamma^0) = \gamma^5$

$$\Rightarrow \underbrace{\underbrace{\frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}_{= S^k} p_k}_{\vec{S}\cdot\vec{p}} \; \frac{1}{2}(1\mp\gamma 5)\psi = \frac{1}{2} \underbrace{p^0}_{\mathrm{sgn}(p^0)\,|\vec{p}|} \gamma^5 \, \frac{1}{2}(1\mp\gamma 5)\psi$$

$$\underbrace{\frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}}_{\text{helicity operator}} \quad \frac{1}{2}(1\mp\gamma^5)\psi = \frac{1}{2}(\operatorname{sgn} p^0) \underbrace{\underbrace{\gamma^5\frac{1}{2}(1\mp\gamma^5)\psi}_{\mp\frac{1}{2}(1\mp\gamma^5)\psi}}_{\mp\frac{1}{2}\operatorname{sgn} p^0}$$

helicity = chirality for these massless particles

Note: For massive spin 1/2 particle helicity and chirality are different quantities

6.10 Charge Conjugation:

investigate another discrete symmetry

$$\psi(x) \longrightarrow \psi^*(x)$$

$$\psi_c'(x) = \underbrace{C}_{=?}\ \bar{\psi}^T(x) = C\gamma^0\psi^*(x)$$

$$\bar{\psi}(x) = \psi^+(x)\gamma^0, \quad \bar{\psi}^T(x) = (\gamma^0)^T(\psi^+(x))^T = \gamma^0\psi^*(x)$$

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \,|\, + \quad \longrightarrow \quad (i\gamma^\mu\partial_\mu - m)\psi_c'(x) = 0$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$-i\,\partial_\mu \psi^+(x)\gamma^{\mu+} - m\psi^+(x) = 0$$

$$\Big\downarrow\ \psi^+(x) = \overline{\psi}(x)\gamma^0$$

$$-i\,\partial_\mu \overline{\psi}(x)\gamma^0\gamma^{\mu+} - m\,\overline{\psi}(x)\gamma^0 = 0 \quad \Big)\cdot\gamma^0$$

$$i\,\partial_\mu \overline{\psi}(x)\underbrace{\gamma^0(\gamma^{\mu})^+\gamma^0}\ + m\,\overline{\psi}(x) = 0$$

$$= \begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0\end{pmatrix}^+\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix} = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0\end{pmatrix} = \gamma^\mu$$

$$\boxed{+i\,\partial_\mu \overline{\psi}(x)\gamma^\mu + m\,\overline{\psi}(x) = 0}$$

$$(i\gamma^\mu\partial_\mu - m)\, C\, \overline{\psi}^T(x) = 0 \quad \Big|\ C^{-1}\cdot$$

$$(i\,C^{-1}\gamma^\mu C\,\partial_\mu - \underbrace{C^{-1}C}_{=1}\, m)\,\overline{\psi}^T(x) = 0 \quad \Big|\ T$$

$$i\,\partial_\mu\overline{\psi}(x)\,(C^{-1}\gamma^\mu C)^T - m\,\overline{\psi}(x) = 0$$

$$(C^{-1}\gamma^\mu C)^T = -\gamma^\mu$$

$$\boxed{C^{-1}\gamma^\mu C = -(\gamma^\mu)^T}$$

to be solved for C

Solution ansatz: $C = \begin{pmatrix} c & 0 \\ 0 & -c\end{pmatrix}$, $C^{-1} = \begin{pmatrix} c^{-1} & 0 \\ 0 & -c^{-1}\end{pmatrix}$

c: 2×2-matrix

$$\begin{pmatrix} c^{-1} & 0 \\ 0 & -c^{-1}\end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & -c\end{pmatrix} = \ldots = \begin{pmatrix} 0 & -c^{-1}\sigma^\mu c \\ -c^{-1}\tilde{\sigma}^\mu c & 0\end{pmatrix}$$

$$= -\begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0\end{pmatrix}^T = -\begin{pmatrix} 0 & (\tilde{\sigma}^\mu)^T \\ (\sigma^\mu)^T & 0\end{pmatrix}$$

$$\boxed{\begin{aligned} c^{-1}\sigma^\mu c &= +(\tilde{\sigma}^\mu)^T \\ c^{-1}\tilde{\sigma}^\mu c &= +(\sigma^\mu)^T\end{aligned}}$$

$\mu = 0$: $\quad c^{-1}\sigma^0 c = (\sigma^0)^T$, $\quad \mu = k$: $\quad c^{-1}\sigma^k c = -(\sigma^k)^T$

Pauli matrices:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix},\ (\sigma^0)^T = \sigma^0 \quad ; \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix},\ (\sigma^1)^T = \sigma^1$$

$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $(\sigma^2)^T = -\sigma^2$; $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $(\sigma^3)^T = \sigma^3$

Ansatz: $\boxed{c = -i\sigma^2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\boxed{c^+ = c^{-1} = c^T = -c = \boxed{-c^*}}$$

$c^{-1}\sigma^0 c = i\sigma^2\sigma^0(-i\sigma^2) = \sigma^2\sigma^0\sigma^2 = (\sigma^2)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^0 = (\sigma^0)^T$ $\mu = 0$ ✓

$c^{-1}\sigma^1 c = i\sigma^2\sigma^1(-i\sigma^2) = \sigma^2\sigma^1\sigma^2 = -\sigma^1(\sigma^2)^2 = -\sigma^1 = -(\sigma^1)^T$ $\mu = 1$ ✓

$\Rightarrow$ show als $\mu = 2$, $\mu = 3$

Conclusion: $C^+ = C^{-1} = C^{-1} = -C = -C^*$

representation of C in γ-matrices:

$$i\gamma^0\gamma^2 = i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} = C$$

Involutoric property:

$$\psi_c''(x) = C\gamma^0\psi_c'^*(x) = C\gamma^0\left[C\gamma^0\psi^*(x)\right]^* = \underbrace{C\gamma^0 C^*\gamma^0}\psi(x)$$

$$= \underbrace{\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}_{= \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}}\underbrace{\begin{pmatrix} c^* & 0 \\ 0 & -c^* \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}_{= \begin{pmatrix} 0 & c^* \\ -c^* & 0 \end{pmatrix}} = \begin{pmatrix} -cc^* & 0 \\ 0 & -cc^* \end{pmatrix}\psi(x)$$

$= \psi(x)$ ✓

$c^* = -c^{-1}$

Continuity equation:

$$i\gamma^\mu \partial_\mu \psi(x) - m\psi(x) = 0 \xrightarrow{\bar\psi(x)\cdot} i\bar\psi(x)\gamma^\mu\partial_\mu\psi(x) - m\bar\psi(x)\psi(x) = 0$$

$$i\partial_\mu\bar\psi(x)\gamma^\mu + m\bar\psi(x) = 0 \xrightarrow{\cdot\psi(x)} i\partial_\mu\bar\psi(x)\gamma^\mu\psi(x) + m\bar\psi(x)\psi(x) = 0$$

$$\Rightarrow \quad \bar\psi(x)\gamma^\mu\partial_\mu\psi(x) + \partial_\mu\bar\psi(x)\gamma^\mu\psi(x) = \partial_\mu\underbrace{\left[\bar\psi(x)\gamma^\mu\psi(x)\right]}_{\sim\, j^\mu(x)} = 0$$

current density: $j^\mu(x) = K\,\bar\psi(x)\gamma^\mu\psi(x)$

conserved charge: $Q = \int d^3x\, j^0(\vec{x},t) = K\int d^3x\,\underbrace{\psi^+(x)\gamma^0}_{=\bar\psi(x)}\underbrace{\gamma^0}_{=(\gamma^0)^2=1}\psi(x)$

$$\Rightarrow \quad Q = K\int d^3x\, \psi^+(\vec{x},t)\psi(\vec{x},t)$$

$$\bar\psi_c'(x) = \psi_c'^+(x)\gamma^0 = \left(C\gamma^0\psi^*(x)\right)^+\gamma^0 = \underbrace{\psi^T(x)\underbrace{(\gamma^0)^+}_{=\gamma^0}\underbrace{C^+}_{-C}\gamma^0}$$

$$= \psi^T(x)\,C \qquad\qquad = -\gamma^0 C\gamma^0 = \ldots = C$$

charge conjugation of current density:

$$j_c'^\mu(x) = K\,\bar\psi_c'(x)\gamma^\mu\psi_c'(x) = K\,\psi^T(x)\,C\gamma^\mu C\gamma^0\psi^*(x)$$

$$= \left(j_c'^\mu(x)\right)^T = K\,\underbrace{\psi^+(x)\underbrace{(\gamma^0)^T}_{=\gamma^0}}_{=\bar\psi(x)}\underbrace{\left(C\gamma^\mu C\right)^T}_{\gamma^\mu}\psi(x) = j^\mu(x)$$

↑ c number

no sign change

Note: We do not get the expected sign change

$\Rightarrow$ Revisit charge conjugation on level of 2nd quantization yields sign change.

6.11 Time Inversion: "run a movie backwards in time"

$$x = (x^0, x^k) \longrightarrow x'_T = T x = -\tilde{x} = (-x^0, x^k)$$

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T^2 = 1 \iff T^{-1} = T$$

show:
$$T^{-1} L_k T^{-1} = L_k \iff [T, L_k]_- = 0$$

$$T^{-1} M_k T^{-1} = -M_k \iff [T, M_k]_+ = 0$$

Side consideration: Schrödinger

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta\right) \psi(\vec{x}, t) = 0$$

time-inverted wave function $\psi'_T(\vec{x}, t) = \psi^*(\vec{x}, -t)$

$$\Rightarrow \left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta\right) \psi'_T(\vec{x}, t) = 0$$

by analogy: time inversion for Dirac spinor:

$$\psi(x) \xrightarrow{T} \boxed{\psi'_T(x) = \underbrace{D(T)}\, \psi^*(-\tilde{x})}$$

representation matrix for time inversion for Dirac spinors

Expectations:

$D(T)^2 = 1$, $D(T)^{-1} D(L_R) D(T) = D(L_R)$, $D(T)^{-1} D(M_R) D(M_R) = -D(M_R)$

$$(i\gamma^\mu\partial_\mu - m)\psi(x) \xrightarrow{T} (i\gamma^\mu\partial_\mu - m)\psi'_T(x) = 0$$

$$(i\gamma^\mu\partial_\mu - m)\, D(T)\,\psi^*(-\tilde{x}) = 0 \quad \Big|\ D(T)^{-1}\cdot$$

$$\Big[i\, D(T)^{-1}\gamma^\mu D(T)\,\partial_\mu - m \underbrace{D(T)^{-1}D(T)}_{=1}\Big]\psi^*(-\tilde{x}) = 0 \quad \Big|\ *$$

$$-i\big[D(T)^{-1}\gamma^\mu D(T)\big]^*\partial_\mu\,\psi(-\tilde{x}) - m\,\psi(-\tilde{x}) = 0$$

$$\xrightarrow{x\to-\tilde{x}} \Big(-i\underbrace{\gamma^\mu\tilde{\partial}_\mu}_{=\tilde{\gamma}^\mu\partial_\mu} - m\Big)\psi(-\tilde{x}) = 0$$

$$\big[D(T)^{-1}\gamma^\mu D(T)\big]^* = \tilde{\gamma}^\mu \Rightarrow D(T)^{-1}\gamma^\mu D(T) = (\tilde{\gamma}^\mu)^*$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \Rightarrow (\gamma^0)^* = \gamma^0,\ (\gamma^1)^* = \gamma^1,\ (\gamma^2)^* = -\gamma^2,\ (\gamma^3)^* = \gamma^3$$

$$\Rightarrow \ldots \Rightarrow (\gamma^\mu)^* = (\tilde{\gamma}^\mu)^T \Rightarrow (\gamma^\mu)^\dagger = \tilde{\gamma}^\mu$$

$\Rightarrow D(T)^{-1}\gamma^\mu D(T) = (\gamma^\mu)^T$ has to solved for $D(T)$

Remark: $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$

$$D(T)^{-1}\gamma^\mu D(T) = -C^{-1}\gamma^\mu C \quad \Big|\ C\cdot \quad \cdot C^{-1}$$

$$\underbrace{C\, D(T)^{-1}}_{=(D(T)\, C^{-1})^{-1}} \gamma^\mu D(T) C^{-1} = -\gamma^\mu$$

$$[D(T) C^{-1}]^{-1} \gamma^\mu [D(T) C^{-1}] = -\gamma^\mu$$

Solution: $D(T)\, C^{-1} = -i\gamma^5 \Rightarrow$ your exercise

Note: Technically here all 3 discrete transformations are connected.

$\Rightarrow D(T) = -i\gamma^5 C = \dots = i\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$

Properties: $D(T) = D(T)^{-1} = D(T)^{+} = -D(T)^{*} = -D(T)^{T}$

Here T is not involutoric in spinor space:

$$\psi_T''(x) = D(T)\, \psi_T'^{*}(-\tilde{x}) = D(T) \{ D(T)\, \psi^{*}(-\tilde{x}) \}^{*} \Big|_{x \to -\tilde{x}} = \underbrace{D(T) D(T)^{*}}_{= D(T) D(T)^{-1} = -1} \psi(x)$$

$= -\psi(x)$

$\Rightarrow$ property of Dirac spinors

For you to show:

$D(T)^{-1} D(L_\Lambda) D(T) = D(L_\Lambda)^{*} \neq D(L_\Lambda)$ Expectations

$D(T)^{-1} D(M_\Lambda) D(T) = -D(M_\Lambda)^{*} \neq -D(M_\Lambda)$ not fulfilled

In second quantization time inversion is defined with an anti-linear operator

anti-linear

$$\hat{J}\left(\alpha_1 \hat{O}_1 + \alpha_2 \hat{O}_2\right) = \alpha_1^{*} \hat{J}(\hat{O}_1) + \alpha_2^{*} \hat{J}(\hat{O}_2)$$

2nd quantized operators

c numbers