6.11 Dirac Representation:

so far (chiral) Weyl representation ≙ adequate for Lorentz invariance due to the block diagonal form of

$$D(\Lambda) = \begin{pmatrix} D^{(1/2,0)}(\Lambda) & 0 \\ 0 & D^{(0,1/2)}(\Lambda) \end{pmatrix}$$

now (standard) Dirac representation ≙ adequate for non-relativistic limit, i.e. upper Weyl spinor survives, lower Weyl spinor negligible

$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ diagonal — γ^5_D no longer diagonal

$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ non-diagonal — γ^0_D diagonal

$$\psi_D(x) = S_D\,\psi(x), \quad S_D = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}, \quad S_D^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} = S_D^T$$

S_D orthonormal, i.e. unitary

$$\bar{\Psi}_D(x) = \Psi_D^+(x)\gamma^0 = \Psi^+(x) S_D^+ \gamma^0 = \bar{\Psi}(x) \underbrace{\gamma^0 S_D^+ \gamma^0}_{= S_D^{-1}}$$

Dirac matrices in Dirac representation

$$\gamma_D^0 = S_D \gamma S_D^{-1} = \dots = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{diagonal}$$

$$\gamma_D^k = S_D \gamma^k S_D^{-1} = \dots = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

chirality matrix:

$$\gamma_D^5 = S_D \gamma^5 S_D^{-1} = \dots = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{not diagonal}$$

generators of

- rotations: $D(L_k)_D = S_D D(L_k) S_D^{-1} = \dots = \frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$

  → Weyl and Dirac representation are identical

- boost: $D(M_k)_D = S_D D(M_k) S_D^{-1} = \dots = \frac{i}{2}\begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix}$

6.12 Non-Relativistic Limit

$\hat{=}$ historic approach of Paul Dirac

$$(i\gamma^\mu \partial_\mu - m)\,\Psi(x) = 0 \quad \Big|\; S_D \cdot \qquad \cdot S_D^{-1}$$

$$\Rightarrow \left(i\gamma_D^\mu \partial_\mu - m\right)\Psi_D(x) = 0$$

separate temporal and spatial degrees of freedom

$$\underbrace{i\gamma_D^0 \frac{1}{c}}\frac{\partial}{\partial t}\Psi_D(\vec{x},t) + i\gamma_D^k \partial_k \Psi_D(\vec{x},t) - m\,\Psi_D(\vec{x},t) = 0 \quad \Big|\; \hbar c\,\gamma_D^0 \cdot$$

$$i\hbar\underbrace{(\gamma_D^0)^2}\frac{\partial}{\partial t}\Psi_D(\vec{x},t) + i\hbar c\,\underbrace{\gamma_D^0\gamma_D^k}_{\alpha^k =}\partial_k\Psi_D(\vec{x},t) - \hbar c\, m\,\underbrace{\gamma_D^0}_{\beta =}\Psi_D(\vec{x},t) = 0$$

$$S_D\gamma^0\underbrace{S_D^{-1}S_D}_{=1}\gamma^0 S_D^{-1} = S_D(\gamma^0)^2 S_D^{-1} = 1$$

Schrödinger-like equation

$$i\hbar\frac{\partial}{\partial t}\Psi_D(\vec{x},t) = \hat{H}\,\Psi_D(\vec{x},t)$$

$$\hat{H} = -i\hbar c\,\vec{\alpha}\cdot\vec{\partial} + \hbar c\, m\beta$$

$\beta = \gamma^0_D = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\alpha^k = \gamma^0_D \gamma^k_D = \dots = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$

$[\beta, \beta]_+ = 2 \cdot \tilde{I}$, $\tilde{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ $\qquad [\alpha^k, \alpha^l]_+ = 2 \delta_{kl} \tilde{I}$

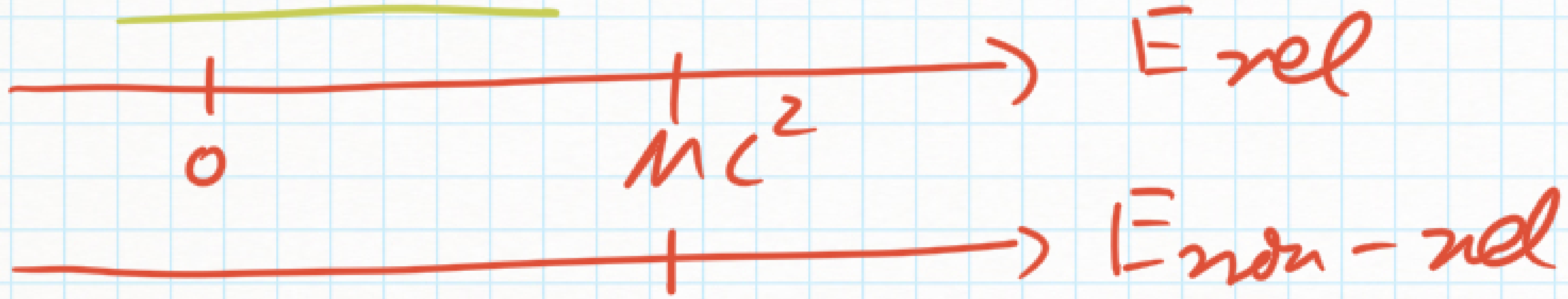
$[\alpha^k, \beta]_+ = \mathcal{O}$, $\mathcal{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Decompose ψ_D into Weyl spinors ζ_D, ξ_D: $\psi_D = \begin{pmatrix} \zeta_D \\ \xi_D \end{pmatrix}$

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \zeta_D(\vec{x},t) \\ \xi_D(\vec{x},t) \end{pmatrix} = -i\hbar c \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \cdot \vec{\nabla} \begin{pmatrix} \zeta_D(\vec{x},t) \\ \xi_D(\vec{x},t) \end{pmatrix} + \hbar c m \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \zeta_D(\vec{x},t) \\ \xi_D(\vec{x},t) \end{pmatrix}$$

$$i\hbar \underline{\frac{\partial \zeta_D(\vec{x},t)}{\partial t}} = -i\hbar c\, \vec{\sigma}\cdot\vec{\nabla}\, \xi_D(\vec{x},t) + \hbar c m\, \zeta_D(\vec{x},t)$$

$$i\hbar \underline{\frac{\partial \xi_D(\vec{x},t)}{\partial t}} = -i\hbar c\, \vec{\sigma}\cdot\vec{\nabla}\, \zeta_D(\vec{x},t) - \hbar c m\, \xi_D(\vec{x},t)$$



$$\psi_D(\vec{x},t) = \begin{pmatrix} \tilde{\zeta}_D(\vec{x},t)\, e^{-\frac{i}{\hbar} mc^2 t} \\ \tilde{\xi}_D(\vec{x},t)\, e^{-\frac{i}{\hbar} mc^2 t} \end{pmatrix} = \begin{pmatrix} \zeta_D(\vec{x},t) \\ \xi_D(\vec{x},t) \end{pmatrix}$$

$$i\hbar \frac{\partial \tilde{\xi}_D(\vec{x},t)}{\partial t} = -i\hbar c(\vec{\sigma}\cdot\vec{\nabla})\tilde{\xi}_D(\vec{x},t) + \underbrace{(\hbar c m - Mc^2)}_{=0}\tilde{\xi}_D(\vec{x},t) \qquad (1)$$

$$\cancel{i\hbar \frac{\partial \tilde{\xi}_D(\vec{x},t)}{\partial t}} = -i\hbar c(\vec{\sigma}\cdot\vec{\nabla})\tilde{\xi}_D(\vec{x},t) + \underbrace{(-\hbar c m - Mc^2)}_{=-2Mc^2}\tilde{\xi}_D(\vec{x},t) \Leftarrow \checkmark \qquad (2)$$

$m = \frac{Mc}{\hbar}$; each Weyl spinor fulfills Klein-Gordon equation

large term

Non-relativistic limit: $c \to \infty$

$$\Rightarrow \left| i\hbar \frac{\partial \tilde{\xi}_D(\vec{x},t)}{\partial t} \right| \ll Mc^2\,\tilde{\xi}_D(\vec{x},t)$$

small expansion parameter $\frac{E_{kin}}{Mc^2}$

$\Rightarrow$ adiabatic elimination of $\tilde{\xi}_D(\vec{x},t)$

now $\tilde{\xi}_D(\vec{x},t) = -\frac{i\hbar c}{2Mc^2}(\vec{\sigma}\cdot\vec{\nabla})\tilde{\xi}_D(\vec{x},t)$ just $\qquad (2')$

$\tilde{\xi}_D$ is no longer a dynamical degree of freedom

Adiabatic elemination occurs quite often in physics:

- Born-Oppenheimer approximation in molecular physics

nuclei	electrons
slow motion	fast motion

$\Rightarrow$ motion of nuclei is adiabatically eliminated

smallness parameter: $\dfrac{m_{\text{electron}}}{m_{\text{nuclei}}}$

- Laser physics (semi-classical laser theory):

electric field

slow: u

enslaving

polarization

population inversion

fast: s

mathematical theorem: center manifold

$s = s(u)$

$\Rightarrow$ self-organization theory (Hermann Haken)

Resulting evolution equation for $\tilde{\xi}_D$:

$$i\hbar \frac{\partial}{\partial t}\tilde{\xi}_D(\vec{x},t) = \underbrace{(-i\hbar c)\frac{-i\hbar c}{2mc^2}}_{=-\frac{\hbar^2}{2m}} \underbrace{(\vec{\sigma}\cdot\vec{\nabla})(\vec{\sigma}\cdot\vec{\nabla})}_{\Delta = \frac{1}{2}\underbrace{[\sigma^k,\sigma^l]_+}_{=2\delta_{kl}} \underbrace{\partial_k \partial_l}_{\text{symm. in } k,l}} \tilde{\xi}_D(\vec{x},t)$$

$\Rightarrow$ upper Weyl spinor $\tilde{\xi}_D$ follows a Schrödinger equation

Dirac action:

$$A = \frac{A}{c}\int d^4x\ \bar{\psi}_D(x)\left(i\gamma^\mu_D \partial_\mu - m\right)\psi_D(x)$$

not yet determined

$$= \frac{A}{c}\int_{=c\,dt\,d^3x} d^4x \left\{ i \underbrace{\bar{\psi}_D \gamma^0_D}_{=\psi_D^+(\gamma^0_D)^2=\psi_D^+} \frac{1}{c}\frac{\partial \psi_D}{\partial t} + i\underbrace{\bar{\psi}_D \vec{\gamma}_D}_{\psi_D^+ \underbrace{\gamma^0_D\vec{\gamma}_D}_{=\vec{\alpha}}}\cdot\vec{\nabla}\psi_D - m\underbrace{\bar{\psi}_D}_{\psi_D^+\beta}\psi_D \right\}$$

$$\psi_D = \begin{pmatrix} \xi_D \\ \eta_D \end{pmatrix} = \begin{pmatrix} \tilde\xi_D e^{-\frac{i}{\hbar}mc^2 t} \\ \tilde\eta_D e^{-\frac{i}{\hbar}mc^2 t} \end{pmatrix}, \quad \psi_D^+ = \cdots$$

$$S = A\int dt \int d^3x \left\{ \frac{i}{c}\left[\tilde\xi_D^+ \frac{\partial \tilde\xi_D}{\partial t} + \cancel{\tilde\eta_D^+ \frac{\partial \tilde\eta_D}{\partial t}}\right] + 2\frac{mc}{\hbar}\left[\cancel{\tilde\xi_D^+ \tilde\xi_D} + \tilde\eta_D^+ \tilde\eta_D\right] \right.$$

$$\left. + i\left[\tilde\xi_D^+ \vec\sigma \vec\nabla \tilde\eta_D + \tilde\eta_D^+ \vec\sigma \vec\nabla \tilde\xi_D\right] - \frac{mc}{\hbar}\left[\cancel{\tilde\xi_D^+ \tilde\xi_D} - \cancel{\tilde\eta_D^+ \tilde\eta_D}\right] \right\}$$

$m \to \frac{mc}{\hbar}$

non-relativistic: $\frac{\partial \tilde\eta_D}{\partial t} = 0$, $\tilde\eta_D = -\frac{i\hbar c}{2mc^2}\vec\sigma\vec\nabla\tilde\xi_D$

$$\cancel{(1)\ 2\frac{mc}{\hbar}\left(\frac{-i\hbar c}{2mc^2}\right)^2 (\vec\nabla\tilde\xi_D^+ \cdot \vec\sigma^+)(\vec\sigma\cdot\vec\nabla\tilde\xi_D)}$$

$$(2)\ i\frac{-i\hbar c}{2mc^2}\tilde\xi_D^+ (\vec\sigma\vec\nabla)(\vec\sigma\vec\nabla)\tilde\xi_D = \frac{\hbar}{2mc}\tilde\xi_D^+ \underbrace{(\vec\sigma\cdot\vec\nabla)(\vec\sigma\vec\nabla)}_{=\Delta}\tilde\xi_D$$

$$\cancel{(3)\ i\frac{i\hbar c}{2mc^2}(\vec\nabla\tilde\xi_D^+)\vec\sigma^+ \cdot \vec\sigma\vec\nabla\tilde\xi_D}$$

$A = \hbar c$

$$S = A\int dt\, d^3x \left[\frac{i}{c}\tilde\xi_D^+ \frac{\partial\tilde\xi_D}{\partial t} + \frac{\hbar}{2mc}\tilde\xi_D^+ \Delta \tilde\xi_D\right] \overset{!}{=} \int dt\, d^3x \left[i\hbar\tilde\xi_D^+ \frac{\partial\tilde\xi_D}{\partial t} + \frac{\hbar^2}{2m}\tilde\xi_D^+ \Delta\tilde\xi_D\right]$$

Conclusion:

$\mathcal{L} = \overline{\Psi}\left(i\hbar c\, \overline{\Psi}\, \gamma^\mu \partial_\mu \Psi - mc^2 \Psi\right)$

6.13 Plane Waves:

Dirac equation in Weyl representation: $\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right)\Psi(x) = 0$

Here: group theoretically inspired solution

- go into rest frame, solve there Dirac equation
- boost rest frame solution to a uniformly moving reference frame

6.13.1 Rest Frame:

$\Psi_R(x) = \Psi(t) \Rightarrow \left(i\gamma^0 \frac{\partial}{\partial t} - \frac{mc^2}{\hbar}\right)\Psi(t) = 0 \quad \Big| \left(-i\gamma^0\right)\frac{\partial}{\partial t} - \frac{mc^2}{\hbar}$

$\left(-i\gamma^0 \frac{\partial}{\partial t} - \frac{mc^2}{\hbar}\right)\left(i\gamma^0 \frac{\partial}{\partial t} - \frac{mc^2}{\hbar^2}\right)\Psi(t) = \Big[\underbrace{(\gamma^0)^2}_{=1} \frac{\partial^2}{\partial t^2} + \left(\frac{mc^2}{\hbar}\right)^2\Big]\Psi(t) = 0$

$$\Psi(t) = \Psi\, e^{\mp \frac{i}{\hbar} mc^2 t}$$

Algebraic equation for spinor amplitude

$$(\pm \gamma^0 - I)\,\Psi = 0, \quad \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$+ : \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} \Psi = 0, \qquad - : \begin{pmatrix} -I & -I \\ -I & -I \end{pmatrix} \Psi = 0$$

assumption: $\chi(\pm \frac{1}{2})$ two biorthonormal bispinors

$$\chi^{+}(\lambda)\,\chi(\lambda') = \delta_{\lambda\lambda'}$$

$$+ : \Psi^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi(1/2) \\ \chi(1/2) \end{pmatrix}, \quad \Psi^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi(-1/2) \\ \chi(-1/2) \end{pmatrix}$$

construct charge conjugated bispinor

$$\chi^{c}(\pm \tfrac{1}{2}) = c\,\chi^{*}(\pm \tfrac{1}{2}), \quad c = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \chi^{c}(\lambda)^{+}\,\chi^{c}(\lambda') = \delta_{\lambda\lambda'}$$

$\bullet$: $\psi^{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^c(1/2) \\ -\chi^c(1/2) \end{pmatrix}$, $\psi^{(4)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^c(-1/2) \\ -\chi^c(-1/2) \end{pmatrix}$

$\psi^{(3)}$ and $\psi^{(4)}$ charge conjugated solution of $\psi^{(1)}, \psi^{(2)}$:

$$\overline{\psi}^{(1,2)} = \psi^{(1,2)+} \gamma^0 = \frac{1}{\sqrt{2}} \left(\chi^+(\lambda), \chi^+(\lambda) \right)$$

$$\psi_c^{(1,2)} = C \, \overline{\psi}^{(1,2)T} = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^*(\lambda) \\ \chi^*(\lambda) \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} c\,\chi^*(\lambda) \\ -c\,\chi^*(\lambda) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^c(\lambda) \\ -\chi^c(\lambda) \end{pmatrix} = \psi^{(3,3)} \checkmark$$