6.14 Plane Waves:

Solution of Dirac equation in Weyl representation:

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right)\psi(x) = 0$$

6.14.1 Rest Frame:

$$\psi_R(t) = \underbrace{\psi}_{\text{spinor}}\, e^{\mp i \frac{mc^2}{\hbar} t}$$

$$\psi^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi(1/2)\\ \chi(1/2)\end{pmatrix}, \quad \psi^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi(-1/2)\\ \chi(-1/2)\end{pmatrix}$$

2-spinors: $\chi^+(\lambda)\,\chi(\lambda') = \delta_{\lambda\lambda'}$

concrete representation discuss later

$$\psi^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi^c(1/2)\\ -\chi^c(1/2)\end{pmatrix}, \quad \psi^{(4)} = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi^c(-1/2)\\ -\chi^c(-1/2)\end{pmatrix}$$

$$\chi^c(\lambda) = c\,\chi^*(\lambda) \implies \psi_c^{(1,2)} = C\,\bar{\psi}^{(1,2)T} = \psi^{(3,4)}$$

6.14.2 Boost to Uniformly Moving Reference Frame:

$$\psi^{(1,2)}\, e^{-\frac{i}{\hbar} mc^2 t} \xrightarrow{B} \psi_{\vec{p}}^{(1,2)}(x) = \psi_{\vec{p}}^{(1,2)}\, e^{-\frac{i}{\hbar} p x}$$

$$\psi^{(3,4)}\, e^{+\frac{i}{\hbar} mc^2 t} \xrightarrow{B} \psi_{\vec{p}}^{(3,4)}(x) = \psi_{\vec{p}}^{(3,4)}\, e^{+\frac{i}{\hbar} p x}$$

momentum?

$$p^\mu_R\, p_{R\mu} = m^2c^2 = p_\mu p^\mu = (p^0)^2 - \vec{p}^{\,2} \Rightarrow p^0 = \frac{E_{\vec{p}}}{c} = \sqrt{\vec{p}^{\,2}c^2 + m^2c^4}$$

$$\psi^{(r)}_{\vec{p}} = D(B)\ \psi^{(r)} \qquad r = 1,2,3,4$$

$$D(B) = \begin{pmatrix} D^{(1/2,0)}(B) & 0 \\ 0 & D^{(0,1/2)}(B) \end{pmatrix} = \begin{pmatrix} e^{-\vec{\sigma}\cdot\vec{\zeta}/2} & 0 \\ 0 & e^{+\vec{\sigma}\cdot\vec{\zeta}/2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{p\sigma + mc}{\sqrt{2mc(p^0+mc)}} & 0 \\ 0 & \frac{p\tilde{\sigma} + mc}{\sqrt{2mc(p^0+mc)}} \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} \sqrt{\frac{p\sigma}{mc}} & 0 \\ 0 & \sqrt{\frac{p\tilde{\sigma}}{mc}} \end{pmatrix}$$

short-cut notation

$$\psi^{(1,2)}_{\vec{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{mc}}\ \chi(\pm\frac{1}{2}) \\ \sqrt{\frac{p\tilde{\sigma}}{mc}}\ \chi(\pm 1/2) \end{pmatrix}, \qquad \psi^{(3,4)}_{\vec{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{mc}}\ \chi^c(\pm 1/2) \\ -\sqrt{\frac{p\tilde{\sigma}}{mc}}\ \chi^c(\pm\frac{1}{2}) \end{pmatrix}$$

side calculation:

$$(p\sigma)(p\tilde{\sigma}) = \underbrace{p_\mu p_\nu}_{\text{sym. in } \mu,\nu} \sigma^\mu \tilde{\sigma}^\nu = \frac{1}{2} p_\mu p_\nu (\sigma^\mu\tilde{\sigma}^\nu + \sigma^\nu\tilde{\sigma}^\mu) \underset{\uparrow}{=} \frac{1}{2} p_\mu p_\nu\, 2g^{\mu\nu}\, \mathbb{I}$$

previously shown

$$= p^2\, \mathbb{I} = (mc)^2\, \mathbb{I}$$

$$\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi^{(1,2)}_{\vec{p}}(x) = 0 \Rightarrow (\gamma^\mu p_\mu - mc)\,\psi^{(1,2)}_{\vec{p}} = 0$$

$$\begin{pmatrix} 0 & p\sigma \\ p\tilde{\sigma} & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{mc}}\ \chi(\pm 1/2) \\ \sqrt{\frac{p\tilde{\sigma}}{mc}}\ \chi(\pm 1/2) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} p\sigma\sqrt{\frac{p\tilde{\sigma}}{mc}}\ \chi(\pm 1/2) \\ p\tilde{\sigma}\sqrt{\frac{p\sigma}{mc}}\ \chi(\pm 1/2) \end{pmatrix}$$

$$= \frac{mc}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{mc}} \sqrt{\frac{p\sigma}{mc}\frac{p\tilde{\sigma}}{mc}}\, \chi(\pm 1/2) \\ \sqrt{\frac{p\tilde{\sigma}}{mc}} \sqrt{\frac{p\tilde{\sigma}}{mc}\frac{p\sigma}{mc}}\, \chi(\pm 1/2) \end{pmatrix} = \begin{pmatrix} mc\,I & 0 \\ 0 & mc\,I \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{mc}}\,\chi(\pm 1/2) \\ \sqrt{\frac{p\tilde{\sigma}}{mc}}\,\chi(\pm 1/2) \end{pmatrix} \checkmark$$

analogous : $\psi^{(3,4)}_{\vec{p}}$

$\psi^{(3,4)}_{\vec{p}}$ is charge conjugation of $\psi^{(1,2)}_{\vec{p}}$:

$$\overline{\psi}^{(1,2)}_{\vec{p}} = \psi^{(1,2)+}_{\vec{p}}\,\gamma^0 = \frac{1}{\sqrt{2}} \left(\chi^+(\pm\tfrac{1}{2})\sqrt{\frac{p\sigma}{mc}},\ \chi^+(\pm\tfrac{1}{2})\sqrt{\frac{p\tilde{\sigma}}{mc}} \right) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \left(\chi^+(\pm\tfrac{1}{2})\sqrt{\frac{p\tilde{\sigma}}{mc}},\ \chi^+(\pm\tfrac{1}{2})\sqrt{\frac{p\sigma}{mc}} \right)$$

property: $\quad c^{-1}\sigma^\mu c = (\tilde{\sigma}^\mu)^T$, see previous lecture

$$\Rightarrow \quad c(\sigma^\mu)^T c^{-1} = \tilde{\sigma}^\mu, \quad c(\tilde{\sigma}^\mu)^T c^{-1} = \sigma^\mu$$

$f(\sigma^\mu)$ with a Taylor series and $f(0)=0$

$$c\, f(\sigma^\mu)^T c^{-1} = f(\tilde{\sigma}^\mu) \qquad c\, f(\tilde{\sigma}^\mu)^T c^{-1} = f(\sigma^\mu)$$

$$\psi^{(1,2)}_{\vec{p}\,C} = C\,\overline{\psi}^{(1,2)T} = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\tilde{\sigma}}{mc}}^T \chi^*(\pm 1/2) \\ \sqrt{\frac{p\sigma}{mc}}^T \chi^*(\pm 1/2) \end{pmatrix}$$

c^{-1} $\quad c$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{mc}}\,\chi^c(\pm 1/2) \\ \sqrt{\frac{p\tilde{\sigma}}{mc}}\,\chi^c(\pm\tfrac{1}{2}) \end{pmatrix} = \psi^{(3,4)}_{\vec{p}}$$

6.14.3 Orthonormality Relations:

Lecture notes show by different cases:

$$\psi^{(r)}_{\varepsilon_r \vec{p}} + \psi^{(r')}_{\varepsilon_{r'} \vec{p}} = \frac{E_{\vec{p}}}{mc^2}\,\delta_{r,r'}; \qquad \varepsilon_r = \begin{cases} +1; & r = 1,2 \\ -1; & r = 3,4 \end{cases}$$

$$\psi^{(r)}_{\vec{p}}(\vec{x},t) = \psi^{(r)}_{\vec{p}}\, e^{-\frac{i}{\hbar}\varepsilon_r\left(E_{\vec{p}}t - \vec{p}\vec{x}\right)}$$

$$\int d^3x\, \psi^{(r)+}_{\vec{p}}(\vec{x},t)\, \psi^{(r')}_{\vec{p}'}(\vec{x},t) = \underbrace{\psi^{(r)}_{\vec{p}} + \psi^{(r')}_{\vec{p}'}}_{\frac{E_{\vec{p}}}{mc^2}\delta_{rr'}}\, e^{\frac{i}{\hbar}\left(\varepsilon_{r'}\underbrace{E_{\vec{p}'}}_{=E_{\vec{p}}} - \varepsilon_r E_{\vec{p}}\right)t}$$

$$(2\pi\hbar)^3\,\delta\big(\underbrace{\varepsilon_r\vec{p} - \varepsilon_{r'}\vec{p}'}_{=0}\big)$$

$$\vec{p}' = \varepsilon_r \varepsilon_{r'} \vec{p}$$

$$= \frac{E_{\vec{p}}}{mc^2}(2\pi\hbar)^3\,\delta(\vec{p}-\vec{p}')\,\delta_{r,r'}$$

normalised spinal plane waves:

$$\psi^{(r)}_{\vec{p}}(\vec{x},t) = \sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec{p}}}}\;\psi^{(r)}_{\vec{p}}\, e^{-\frac{i}{\hbar}\varepsilon_r\left(E_{\vec{p}}t - \vec{p}\vec{x}\right)}$$

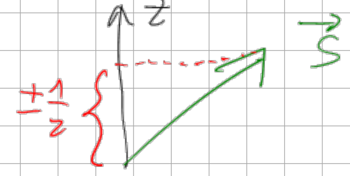
$$\int d^3x\, \psi^{(r)+}_{\vec{p}}(\vec{x},t)\,\psi^{(r')}_{\vec{p}'}(\vec{x},t) = \delta_{r,r'}\,\delta(\vec{p}-\vec{p}')$$

Note: plane waves in Dirac representation, see exercises with Foldy–Wouthuysen transformation.

6.15 Helicity spinors:

so far $\chi(\lambda)$ were not specified, now two particular choices.

6.15.1 Rest Frame:



spin is quantized with respect to z direction

choice for an orthonormal basis

$$\chi\left(+\tfrac{1}{2}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi\left(-\tfrac{1}{2}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

eigenvectors of generator $D(L_3) = \sigma^3/2$:

$$\frac{1}{2}\sigma^3 \, \chi\left(\pm\tfrac{1}{2}\right) = \pm\frac{1}{2}\, \chi\left(\pm\tfrac{1}{2}\right)$$

charge conjugation:

$$\left.\begin{aligned} \chi^c\left(+\tfrac{1}{2}\right) &= c\,\chi^*\left(+\tfrac{1}{2}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \chi^c\left(-\tfrac{1}{2}\right) &= c\,\chi^*\left(-\tfrac{1}{2}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{aligned}\right\} \quad \frac{1}{2}\sigma^3 \chi^c\left(\pm\tfrac{1}{2}\right) = \mp\frac{1}{2}\chi^c\left(\pm\tfrac{1}{2}\right)$$

$\chi(\lambda) \longleftrightarrow \chi^c(\lambda)$: eigenvalues to exchanged

6.15.2 Helicity operator:

analogy to Maxwell chapter: how to construct polarisation of photons

$$D(\vec{L}) = \frac{1}{2}\vec{\sigma}:$$

$$h(\vec{p}) = \frac{D(\vec{L})\cdot\vec{p}}{|\vec{p}|} = \frac{\vec{\sigma}\vec{p}}{2p} = \frac{1}{2p}\Big[p_x \underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{=\sigma_x} + p_y \underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{=\sigma_y} + p_z \underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{=\sigma_z}\Big]$$

$$= \frac{1}{2p}\begin{pmatrix}p_z & p_x - ip_y\\ p_x + ip_y & -p_z\end{pmatrix}$$

helicity spinors = eigenspinors of helicity operator

$$\boxed{h(\vec{p})\,\chi_{\underbrace{h}_{\text{helicity}}}(\vec{p}, \pm\tfrac{1}{2}) = \pm\tfrac{1}{2}\,\chi_h(\vec{p}, \pm\tfrac{1}{2})}$$

special case: $\vec{p} = p\vec{e}_z \Rightarrow h(p\vec{e}_z) = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{1}{2}\sigma^3$

$$\chi_h(p\vec{e}_z, \pm\tfrac{1}{2}) = \chi(\pm\tfrac{1}{2})$$

6.15.3 Uniformly moving rest frame:

$$\vec{p} = p\begin{pmatrix}\sin\vartheta\,\cos\varphi\\ \sin\vartheta\,\sin\varphi\\ \cos\vartheta\end{pmatrix}, \quad \underbrace{R(\vartheta,\varphi)} = \underbrace{R_z(\varphi)R_y(\vartheta)} \overset{\text{Maxwell Lecture}}{=} \begin{pmatrix}\cos\vartheta\,\cos\varphi & -\sin\varphi & \boxed{\sin\vartheta\cos\varphi}\\ \cos\vartheta\,\sin\varphi & \cos\varphi & \boxed{\sin\vartheta\sin\varphi}\\ -\sin\vartheta & 0 & \boxed{\cos\vartheta}\end{pmatrix}$$

$$\vec{p} = R(\vartheta,\varphi)\,\vec{e}_z\,p$$

representation in bi-spinor space:

$$D(R(\vartheta,\varphi)) = \underbrace{D(R_z(\varphi))}\;\underbrace{D(R_y(\vartheta))}$$

$$= e^{-iD(L_3)\varphi} = \cos\left(\frac{\varphi}{2}\right)I - i\sin\left(\frac{\varphi}{2}\right)\sigma^3, \quad = e^{-iD(L_2)\vartheta} = \cos\left(\frac{\vartheta}{2}\right)I - i\sin\frac{\vartheta}{2}\,\sigma^2$$

$$= \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} \qquad = \begin{pmatrix} \cos\vartheta/2 & -\sin\vartheta/2 \\ \sin\vartheta/2 & \cos\vartheta/2 \end{pmatrix}$$

$$D(R(\vartheta,\varphi)) = \begin{pmatrix} \boxed{\begin{matrix}\cos\frac{\vartheta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\vartheta}{2}\, e^{i\varphi/2}\end{matrix}} & \boxed{\begin{matrix}-\sin\frac{\vartheta}{2}\, e^{-i\varphi/2} \\ \cos\frac{\vartheta}{2}\, e^{i\varphi/2}\end{matrix}} \end{pmatrix}$$

$$\chi_h(p\vec{e}_z, \pm\tfrac{1}{2}) = \chi(\pm\tfrac{1}{2}) \xrightarrow{D(R(\vartheta,\varphi))} \chi_h(\vec{p}, \pm\tfrac{1}{2})$$

$$\chi_h(\vec{p}, +\tfrac{1}{2}) = \begin{pmatrix} \cos\frac{\vartheta}{2}\, e^{-i\frac{\varphi}{2}} \\ \sin\frac{\vartheta}{2}\, e^{i\varphi/2} \end{pmatrix} \underset{\vec{p}=p\vec{e}_z}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi(+\tfrac{1}{2}) \quad ✓$$

$$\chi_h(\vec{p}, -\tfrac{1}{2}) = \begin{pmatrix} -\sin\frac{\vartheta}{2}\, e^{-i\varphi/2} \\ \cos\vartheta/2\, e^{i\varphi/2} \end{pmatrix} \underset{\vec{p}=p\vec{e}_z}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \chi(-\tfrac{1}{2}) \quad ✓$$

charge conjugation:

$$\chi_h^c(\vec{p}, +\tfrac{1}{2}) = \underbrace{c}_{= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \underline{\chi_h^*(+\tfrac{1}{2})} = \begin{pmatrix} -\sin\frac{\vartheta}{2}\, e^{-i\varphi/2} \\ \cos\frac{\vartheta}{2}\, e^{i\varphi/2} \end{pmatrix} \underset{\vec{p}=p\vec{e}_z}{=} \underline{\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \chi^c(+\tfrac{1}{2})} \quad ✓$$

$$\chi_h^c(\vec{p}, -\tfrac{1}{2}) = c\, \chi_h^*(-\tfrac{1}{2}) = \begin{pmatrix} -\cos\frac{\vartheta}{2}\, e^{-i\varphi/2} \\ -\sin\frac{\vartheta}{2}\, e^{i\varphi/2} \end{pmatrix} \underset{\vec{p}=p\vec{e}_z}{=} \underline{\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \chi^c(-\tfrac{1}{2})} \quad ✓$$

Crosscheck: $D(R(\vartheta,\varphi)) \quad \chi^c(\pm\tfrac{1}{2}) = \chi_h^c(\vec{p}, \pm\tfrac{1}{2})$

helicity operator: $h(\vec{p}) = \frac{1}{2}\begin{pmatrix} \cos\vartheta & \sin\vartheta\, e^{-i\varphi} \\ \sin\vartheta\, e^{i\varphi} & -\cos\vartheta \end{pmatrix}$

$$h(\vec{p})\, \chi_h(\vec{p}, \pm\tfrac{1}{2}) = \pm\tfrac{1}{2}\, \chi_h(\vec{p}, \pm\tfrac{1}{2}) \quad ✓$$

$$h(\vec{p})\, \chi^c_h(\vec{p}, \pm \tfrac{1}{2}) = \mp \tfrac{1}{2}\, \chi^c_h(\vec{p}, \pm \tfrac{1}{2}) \checkmark$$

Consequences for Dirac spinors:

$$\psi^{(1,2)\,3,4}_{\vec{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{mc}}\, \chi^c(\pm\frac{1}{2}) \\ \sqrt{\frac{p\bar{\sigma}}{mc}}\, \chi^c(\pm\frac{1}{2}) \end{pmatrix} \longrightarrow \psi^{(1,2)\,3,4}_{\vec{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{mc}}\, \chi^c_h(\vec{p}, \pm\frac{1}{2}) \\ \sqrt{\frac{p\bar{\sigma}}{mc}}\, \chi^c_h(\vec{p}, \pm\frac{1}{2}) \end{pmatrix}$$

helicity operator for Dirac spinors

$$H(\vec{p}) := \frac{D(\vec{\Sigma})\cdot\vec{p}}{p} = \frac{1}{2p} \begin{pmatrix} \vec{\sigma}\vec{p} & 0 \\ 0 & \vec{\sigma}\vec{p} \end{pmatrix} = \begin{pmatrix} h(\vec{p}) & 0 \\ 0 & h(\vec{p}) \end{pmatrix}$$

$$\Rightarrow \; --- = H(\vec{p})\,\psi^{(r)}_{\vec{p}} = \xi_r\, \psi^{(r)}_{\vec{p}}$$

$$\xi_r = \frac{(-1)^{r+1}}{2}\; ; \; r = 1,2 \quad ; \quad \xi_r = \frac{(-1)^2}{2}\; ; \; r = 3,4$$

6.16 Canonical Field Quantization:

$$A = \frac{1}{c}\int d^4x\, \mathcal{L}, \quad \mathcal{L} = i\hbar\, \bar{\psi}\gamma^\mu \partial_\mu \psi - mc^2 \bar{\psi}\psi$$

$$\pi(\vec{x},t) = \frac{\delta A}{\delta \frac{\partial \psi(\vec{x},t)}{\partial t}} = i\hbar\, \bar{\psi}(\vec{x},t)\gamma^0 = i\hbar\, \psi^+(\vec{x},t) \underbrace{\gamma^0\gamma^0}_{=1} = i\hbar\, \psi^+(\vec{x},t)$$

$$\bar{\pi}(\vec{x},t) = \frac{\delta A}{\delta \frac{\partial \bar{\psi}(\vec{x},t)}{\partial t}} = 0$$

conserved quantities:

$$Q = \int d^3x\ \psi^+(\vec{x},t)\ \psi(\vec{x},t)$$

$$E = \int d^3x\ \psi^+(\vec{x},t)\ \underbrace{H_D(\vec{x},t)}\ \psi(\vec{x},t) \qquad \rightarrow -i\hbar c\,\vec{\alpha}\cdot\vec{\nabla} + mc^2\beta$$

$$\vec{p} = \int d^3x\ \psi^+(\vec{x},t)\ \frac{\hbar}{i}\vec{\nabla}\ \psi(\vec{x},t)$$

$$h = \int d^3x\ \psi^+(\vec{x},t)\ \underbrace{\begin{pmatrix} \vec{\sigma}/2 & 0 \\ 0 & \vec{\sigma}/2 \end{pmatrix}}_{= D(\vec{\Sigma})} \cdot \frac{"\frac{\hbar}{i}\vec{\nabla}"}{\left|\frac{\hbar}{i}\vec{\nabla}\right|}\ \psi(\vec{x},t)$$

canonical quantization:

$$\psi(\vec{x},t),\ \pi(\vec{x},t) = i\hbar\psi^+ \longrightarrow \hat{\psi}(\vec{x},t),\ \hat{\pi}(\vec{x},t) = \underline{i\hbar\,\hat{\psi}^+(\vec{x},t)}$$

with equal-time anti-commutation relation:

$$[\hat{\psi}_\alpha(\vec{x},t), \hat{\psi}_\beta(\vec{x}',t)]_+ = 0 = [\hat{\pi}_\alpha(\vec{x},t), \hat{\pi}_\beta(\vec{x}',t)]_+$$

$$[\hat{\psi}_\alpha(\vec{x},t), \hat{\pi}_\beta(\vec{x}',t)]_+ = i\hbar\,\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}')$$

$$\rightarrow [\hat{\psi}_\alpha(\vec{x},t), \hat{\psi}^+_\beta(\vec{x}',t)]_+ = \delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}')$$

conserved quantities: operators

$$\hat{Q} = \underline{\int d^3x}\ \hat{\psi}^+(\vec{x},t)\ \hat{\psi}(\vec{x},t)$$

$$\hat{H} = \underline{\int d^3x}\ \hat{\psi}^+(\vec{x},t)\ H_D(\vec{x},t)\ \hat{\psi}(\vec{x},t);$$

$$\hat{\vec{P}} = \int d^3x\, \hat{\psi}^+(\vec{x},t)\, \frac{\hbar}{i} \vec{\nabla}\, \hat{\psi}(\vec{x},t)$$

$$\hat{h} = \int d^3x\, \hat{\psi}^+(\vec{x},t) \begin{pmatrix} \vec{\sigma}/2 & 0 \\ 0 & \vec{\sigma}/2 \end{pmatrix} \cdot \frac{\text{"}\frac{\hbar}{i}\vec{\nabla}\text{"}}{\left|\frac{\hbar}{i}\vec{\nabla}\right|}\, \hat{\psi}(\vec{x},t)$$

Heisenberg equation:

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}(\vec{x},t) = [\hat{\psi}(\vec{x},t), \hat{H}]_- = \ldots = H_D(\vec{x})\, \hat{\psi}(\vec{x},t)$$

6.16 Decomposition in Plane Waves:

$$\hat{\psi}(\vec{x},t) = \sum_{r=1}^{4} \int d^3p\; \psi^{(r)}_{\vec{p}}(\vec{x},t)\, \hat{a}^{(r)}_{\vec{p}} \Rightarrow \hat{a}^{(r)}_{\vec{p}} = \int d^3x\, \psi^{(r)+}_{\vec{p}}\, \hat{\psi}(\vec{x},\underline{\underline{t}})$$

$$\hat{\psi}^+(\vec{x},t) = \sum_{r=1}^{4} \int d^3p\; \psi^{(r)+}_{\vec{p}}(\vec{x},t)\, \hat{a}^{(r)+}_{\vec{p}} \Rightarrow \hat{a}^{(r)+}_{\vec{p}} = \int d^3x\, \psi^{(r)}_{\vec{p}}\, \hat{\psi}^+(\vec{x},\underline{\underline{t}})$$

$$[\hat{a}^{(r)}_{\vec{p}}, \hat{a}^{(r')}_{\vec{p}'}]_+ = [\hat{a}^{(r)+}_{\vec{p}}, \hat{a}^{(r')+}_{\vec{p}'}]_+ = 0, \quad [\hat{a}^{(r)}_{\vec{p}}, \hat{a}^{+(r')}_{\vec{p}'}]_+ = \delta_{r,r'}\, \delta(\vec{p}-\vec{p}')$$

$\hat{a}^{(r)}_{\vec{p}}, \hat{a}^{(r)+}_{\vec{p}}$: annihilation (creation) operator for fermionic particles