Status: Second quantization of Dirac

$$\hat{\Psi}(\vec{x},t) = \sum_{r=1}^{4} \int d^3p \; \underbrace{\psi^{(r)}_{\vec{p}}(\vec{x},t)}_{\text{spinorial plane wave}} \; \hat{a}^{(r)}_{\vec{p}}$$

$$[\hat{a}^{(r)}_{\vec{p}}, \hat{a}^{(r')}_{\vec{p}'}]_+ = 0 = [\hat{a}^{(r)+}_{\vec{p}}, \hat{a}^{(r')+}_{\vec{p}'}]_+ , \quad [\hat{a}^{(r)}_{\vec{p}}, \hat{a}^{(r')+}_{\vec{p}'}]_+ = \delta_{r,r'}\,\delta(\vec{p}-\vec{p}')$$

6.18 Second Quantized Operators:

$$\hat{Q} = \int d^3x \; \hat{\Psi}^+(\vec{x},t)\, \hat{\Psi}(\vec{x},t)$$

$$= \sum_{r=1}^{4} \sum_{r'=1}^{4} \hat{a}^{(r)+}_{\vec{p}} \hat{a}^{(r')}_{\vec{p}'} \underbrace{\int d^3x \; \psi^{(r)*}_{\vec{p}}(\vec{x},t)\, \psi^{(r')}_{\vec{p}'}(\vec{x},t)}_{\delta_{r,r'}\,\delta(\vec{p}-\vec{p}')} = \sum_{r=1}^{4} \underbrace{\hat{a}^{(r)+}_{\vec{p}} \hat{a}^{(r)}_{\vec{p}}}_{\text{positive definite}}$$

$$\hat{H} = \int d^3x \; \hat{\Psi}^+(\vec{x},t)\, H_D(\vec{x})\, \hat{\Psi}(\vec{x},t)$$

$$= \sum_{r=1}^{4} \int d^3p \; \varepsilon_r E_{\vec{p}} \; \hat{a}^{(r)+}_{\vec{p}} \hat{a}^{(r)}_{\vec{p}}$$

$$\varepsilon_r = \begin{cases} +1 & ; r = 1,2 \\ -1 & ; r = 3,4 \end{cases}$$

$$E_{\vec{p}} = \sqrt{\vec{p}^2 c^2 + m^2 c^4}$$

$$= \int d^3p\, E_{\vec{p}} \left\{ \sum_{r=1}^{2} \hat{a}^{(r)+}_{\vec{p}} \hat{a}^{(r)}_{\vec{p}} - \sum_{r=3}^{4} \hat{a}^{(r)+}_{\vec{p}} \hat{a}^{(r)}_{\vec{p}} \right\}$$

negative energies

$$\hat{\vec{P}} = \int d^3x \; \hat{\Psi}^+(\vec{x},t)\, \frac{\hbar}{i} \vec{\nabla}\, \hat{\Psi}(\vec{x},t) = \sum_{r=1}^{4} \int d^3p \; \varepsilon_r\, \vec{p}\; \hat{a}^{(r)+}_{\vec{p}} \hat{a}^{(r)}_{\vec{p}}$$

$$= \int d^3p \; \vec{p} \left\{ \sum_{r=1}^{2} \hat{a}^{(r)+}_{\vec{p}} \hat{a}^{(r)}_{\vec{p}} - \sum_{r=3}^{4} \hat{a}^{(r)+}_{\vec{p}} \hat{a}^{(r)}_{\vec{p}} \right\}$$

$$\hat{h} = \int d^3x\, \hat{\Psi}^+(\vec{x},t) \begin{pmatrix} \vec{\sigma}/2 & 0 \\ 0 & \vec{\sigma}/2 \end{pmatrix} \frac{\frac{\hbar}{i}\vec{\nabla}}{\left|\frac{\hbar}{i}\vec{\nabla}\right|}\, \hat{\Psi}(\vec{x},t)$$

$$= \sum_{r=1}^{4} \sum_{r'=1}^{4} \int d^3p \int d^3p'\, \hat{a}^{(r)+}_{\vec{p}}\, \hat{a}^{(r')}_{\vec{p}'} \int d^3x\, \psi^{(r)*}_{\vec{p}}(\vec{x},t)\, \underbrace{H(\vec{p}')\, \psi^{(r')}_{\vec{p}'}(\vec{x},t)}_{= \xi_{r'}\, \psi^{(r')}_{\vec{p}'}(\vec{x},t)}$$

$$\xi_r = \begin{cases} \frac{(-1)^{r+1}}{2} & ; \ r = 1,2 \\ \frac{(-1)^{r}}{2} & ; \ r = 3,4 \end{cases}$$

$$\Rightarrow \hat{h} = \int d^3p\, \frac{1}{2} \left\{ \sum_{r=1}^{2} \underline{(-1)^{r+1}}\, \hat{a}^{(r)+}_{\vec{p}}\, \hat{a}^{(r)}_{\vec{p}} + \sum_{r=3}^{4} \underline{(-1)^{r}}\, \hat{a}^{(r)+}_{\vec{p}}\, \hat{a}^{(r)}_{\vec{p}} \right\}$$

6.19 Dirac Sea:

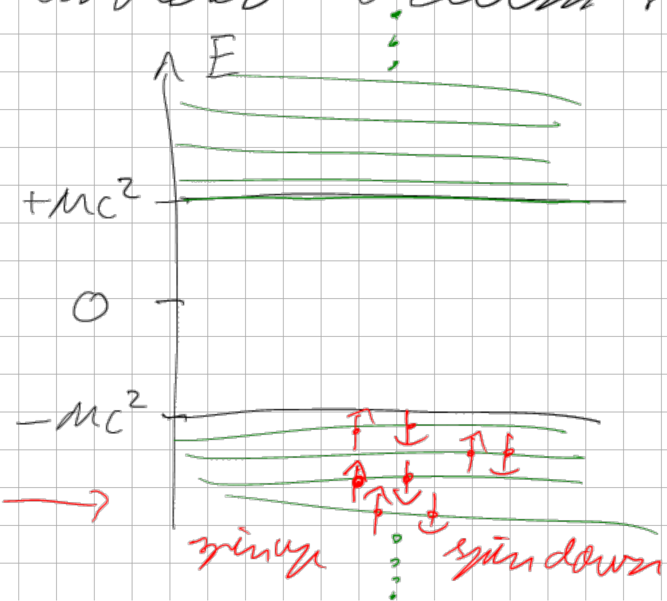
vacuum $|0\rangle_v$: $\hat{a}^{(r)}_{\vec{p}} |0\rangle_v = 0$ for all $\vec{p}, r$

How to physically interpret states with negative energies ?

Paul Dirac (1930): consider instead of $|0\rangle_v$ another vacuum state

$$\underbrace{|0\rangle_P}_{\text{physical}} = \prod_{r=3}^{4} \underbrace{\prod_{\vec{p}}}_{\text{continuum product}} \hat{a}^{(r)+}_{\vec{p}}\, |0\rangle_v$$

Dirac sea $\hat{=}$ all negative energy states are occupied by one electron with spin up / down $\longrightarrow$



Consequence:

$r = 1,2\ ;\ \vec{p}:\qquad \hat{a}^{(r)}_{\vec{p}}\ |0\rangle_P = 0$

$r = 3,4\ ;\ \vec{p}:\qquad \hat{a}^{(r)+}_{\vec{p}}\ |0\rangle_P = 0$ Pauli principle

$\left(\hat{a}^{(r)+}_{\vec{p}}\right)^2 = 0$ due to anticommutativity

Reinterpretation:

$r = 1,2\ ;\ \vec{p}:\ \hat{a}^{(r)}_{\vec{p}},\ \hat{a}^{(r)+}_{\vec{p}}\ \hat{=}$ annihilation / creation operator

$r = 3,4\ ;\ \vec{p}:\ \hat{a}^{(r)}_{\vec{p}},\ \hat{a}^{(r)+}_{\vec{p}}\ \hat{=}$ creation / annihilation "

Dirac hole theory:

$r = 1,2$: particles (e.g. electrons with spin up / down)

$r = 3,4$: antiparticles (e.g. positrons " " " ")

New notation:

- particles: renaming

  creation operators: $\hat{a}^{(1)+}_{\vec{p}} = \hat{b}^{(1)+}_{\vec{p}}\ ,\ \hat{a}^{(2)+}_{\vec{p}} = \hat{b}^{(2)+}_{\vec{p}}$

  annihilation " : $\hat{a}^{(1)}_{\vec{p}} = \hat{b}^{(1)}_{\vec{p}}\ ,\ \hat{a}^{(2)}_{\vec{p}} = \hat{b}^{(2)}_{\vec{p}}$

- antiparticles: exchange of roles

  creation operators: $\hat{a}^{(3)}_{\vec{p}} = \hat{d}^{(1)+}_{-\vec{p}}\ ,\ \hat{a}^{(4)}_{\vec{p}} = \hat{d}^{(2)+}_{-\vec{p}}$

annihilation a: $\hat{a}^{(3)+}_{\vec{p}} = \hat{d}^{(1)}_{\vec{p}}$, $\hat{a}^{(4)+}_{\vec{p}} = \hat{d}^{(2)}_{\vec{p}}$

Anti-commutator algebra does not change:

$$[\hat{b}^{(r)}_{\vec{p}}, \hat{b}^{(r')}_{\vec{p}'}]_+ = [\hat{b}^{(r)}_{\vec{p}}, \hat{d}^{(r')}_{\vec{p}'}]_+ = [\hat{d}^{(r)}_{\vec{p}}, \hat{d}^{(r')}_{\vec{p}'}]_+ = [\hat{b}^{(r)}_{\vec{p}}, \hat{d}^{(r')+}_{\vec{p}'}]_+ = 0$$

$$[\hat{b}^{(r)+}_{\vec{p}}, \ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_ \quad = 0$$

$$[\hat{b}^{(r)}_{\vec{p}}, \hat{b}^{(r')+}_{\vec{p}'}]_+ = [\hat{d}^{(r)}_{\vec{p}}, \hat{d}^{(r')+}_{\vec{p}'}]_+ = \delta_{r,r'}\,\delta(\vec{p}-\vec{p}')$$

physical vacuum:

$$\hat{b}^{(r)}_{\vec{p}}\,|0\rangle_P = 0, \quad \hat{d}^{(r)}_{\vec{p}}\,|0\rangle_P = 0 \quad ; \quad r = 1, 2; \vec{p}$$

Consequences for Hamilton Operator:

$$\hat{H} = \sum_{r=1}^{2} \int d^3p \left\{ E_{\vec{p}}\, \hat{b}^{(r)+}_{\vec{p}}\, \hat{b}^{(r)}_{\vec{p}} - E_{\vec{p}} \underbrace{\hat{d}^{(r)}_{\vec{p}}\, \hat{d}^{(r)+}_{\vec{p}}}_{= -\hat{d}^{(r)+}_{\vec{p}}\hat{d}^{(r)}_{\vec{p}} + \delta(\vec{0})} \right\}$$

$$= \underbrace{\sum_{r=1}^{2} \int d^3p \left\{ E_{\vec{p}}\, \hat{b}^{(r)+}_{\vec{p}}\, \hat{b}^{(r)}_{\vec{p}} + E_{\vec{p}}\, \hat{d}^{(r)+}_{\vec{p}}\, \hat{d}^{(r)}_{\vec{p}} \right\}}_{:\hat{H}: \,=\, \hat{H} - {}_P\langle 0|\hat{H}|0\rangle_P} \underbrace{- \sum_{r=1}^{2} \underbrace{\int d^3p\, E(\vec{p})}_{\infty}\, \underbrace{\delta(\vec{0})}_{\infty}}_{= {}_P\langle 0|\hat{H}|0\rangle_P}$$

particles and antiparticles have positive energies

Consequence for charge operator:

$$\hat{Q} = \sum_{r=1}^{2} \int d^3p \left\{ \hat{b}^{(r)+}_{\vec{p}} \hat{b}^{(r)}_{\vec{p}} + \underbrace{\hat{d}^{(r)}_{\vec{p}} \hat{d}^{(r)+}_{\vec{p}}}_{= -\hat{d}^{(r)+}_{\vec{p}} \hat{d}^{(r)}_{\vec{p}} + \delta(\vec{0})} \right\}$$

$$:\hat{Q}: = \hat{Q} - {}_p\langle 0 | \hat{Q} | 0 \rangle_p = \sum_{r=1}^{2} \int d^3p \left\{ \hat{b}^{(r)+}_{\vec{p}} \hat{b}^{(r)}_{\vec{p}} - \hat{d}^{(r)+}_{\vec{p}} \hat{d}^{(r)}_{\vec{p}} \right\}$$

Consequence for momentum operator:

$$\hat{\vec{P}} = \sum_{r=1}^{2} \int d^3p \left\{ \vec{p}\, \hat{b}^{(r)+}_{\vec{p}} \hat{b}^{(r)}_{\vec{p}} - \vec{p}\, \underbrace{\hat{d}^{(r)}_{\vec{p}} \hat{d}^{(r)+}_{\vec{p}}}_{= -\hat{d}^{(r)+}_{\vec{p}} \hat{d}^{(r)}_{\vec{p}} + \delta(\vec{0})} \right\}$$

$$:\hat{\vec{P}}: = \hat{\vec{P}} - {}_p\langle 0 | \hat{\vec{P}} | 0 \rangle_p$$

$$= \sum_{r=1}^{2} \int d^3p \left\{ \vec{p}\, \hat{b}^{(r)+}_{\vec{p}} \hat{b}^{(r)}_{\vec{p}} + \vec{p}\, \hat{d}^{(r)+}_{\vec{p}} \hat{d}^{(r)}_{\vec{p}} \right\}$$

Consequence for helicity:

$$:\hat{h}: = \hat{h} - {}_p\langle 0 | \hat{h} | 0 \rangle_p = \sum_{r=1}^{2} \int d^3p \, \frac{(-1)^{r+1}}{2} \left\{ \hat{b}^{(r)+}_{\vec{p}} \hat{b}^{(r)}_{\vec{p}} + \hat{d}^{(r)+}_{\vec{p}} \hat{d}^{(r)}_{\vec{p}} \right\}$$

6.20 Propagator:

$$S_{\alpha\beta}(\vec{x}, t; \vec{x}', t') = {}_p\langle 0 | \hat{T} \left(\hat{\psi}_\alpha(\vec{x}, t)\, \bar{\hat{\psi}}_\beta(\vec{x}', t') \right) | 0 \rangle_p$$

$$= \Theta(t-t')\,\hat\psi_\alpha(\vec x,t)\,\hat{\bar\psi}_\beta(\vec x',t') - \Theta(t'-t)\,\hat{\bar\psi}_\beta(\vec x',t')\,\hat\psi_\alpha(\vec x,t)$$

time-ordering for fermions

Aim: equation of motion

$$i\hbar\frac{\partial}{\partial t} S_{\alpha\beta}(\vec x,t;\vec x',t') = i\hbar\,\delta(t-t')\,{}_P\langle 0|\underbrace{\left[\hat\psi_\alpha(\vec x,t),\hat{\bar\psi}_\beta(\vec x',\cancel{t'}t)\right]_+}|0\rangle_P$$

$$= \sum_{\gamma=1}^{4}(\gamma^0)_{\gamma\beta}\underbrace{\left[\hat\psi_\alpha(\vec x,t),\hat\psi^+_\gamma(\vec x',t)\right]_+}_{=\ \delta(\vec x-\vec x')}$$

$$+\,\Theta(t-t')\sum_{\gamma=1}^{4}{}_P\langle 0|\left(-i\hbar c\,\vec\alpha_{\alpha\gamma}\vec\nabla + mc^2\beta_{\alpha\gamma}\right)\hat\psi_\gamma(\vec x,t)\,\hat{\bar\psi}_\beta(\vec x',t')|0\rangle_P$$

$$-\,\Theta(t-t')\,{}_P\langle 0|\hat{\bar\psi}_\beta(\vec x',t')\sum_{\gamma=1}^{4}\left(-i\hbar c\,\vec\alpha_{\alpha\gamma}\vec\nabla + mc^2\beta_{\alpha\gamma}\right)\hat\psi_\gamma(\vec x,t)|0\rangle_P$$

$$i\hbar\frac{\partial}{\partial t} S(\vec x,t;\vec x',t') = \left(-i\hbar c\,\vec\alpha\cdot\vec\nabla + mc^2\beta\right)S(\vec x,t;\vec x',t') + i\hbar\,\gamma^0\,\delta(t-t')\,\delta(\vec x-\vec x')$$

$$\Rightarrow \left(i\hbar\gamma^\mu\partial_\mu - mc\right)S(x,x') = i\hbar\,\delta(x-x')$$

Dirac propagator = Green function of Dirac equation

6.21 Propagator Calculation:

$$\hat{\psi}(\vec{x},t) = \int d^3p \left\{ \sum_{r=1}^{2} \underbrace{\psi^{(r)}_{\vec{p}}(\vec{x},t)}_{=: u^{(r)}_{\vec{p}}(\vec{x},t)} \underbrace{\hat{a}^{(r)}_{\vec{p}}}_{= \hat{b}^{(r)}_{\vec{p}}} + \underbrace{\sum_{r=3}^{4} \psi^{(r)}_{\vec{p}}(\vec{x},t)\, \hat{a}^{(r)}_{\vec{p}}}_{= \sum_{r=1}^{2} v^{(r)}_{\vec{p}}(\vec{x},t)\, \hat{d}^{(r)+}_{\vec{p}}} \right\}$$

$$\hat{\bar{\psi}}(\vec{x},t) = \sum_{r=1}^{2} \int d^3p \left\{ \bar{u}^{(r)}_{\vec{p}}(\vec{x},t)\, \hat{b}^{(r)+}_{\vec{p}} + \bar{v}^{(r)}_{\vec{p}}(\vec{x},t)\, \hat{d}^{(r)}_{\vec{p}} \right\}$$

$$S_{\alpha\beta}(\vec{x},t;\vec{x}',t') = \sum_{r=1}^{2} \sum_{r'=1}^{2} \int d^3p \int d^3p'$$

$$\Big\{ \mathbb{1}(t-t')\; {}_p\langle 0| \left\{ u^{(r)}_{\vec{p}\alpha}(\vec{x},t)\, \hat{b}^{(r)}_{\vec{p}} + \cancel{v^{(r)}_{\vec{p}\alpha}(\vec{x},t)\, \hat{d}^{(r)+}_{\vec{p}}} \right\}$$

$$\cdot \left\{ \bar{u}^{(r')}_{\vec{p}'\beta}(\vec{x}',t')\, \hat{b}^{(r')+}_{\vec{p}'} + \cancel{\bar{v}^{(r')}_{\vec{p}'\beta}(\vec{x}',t')\, \hat{d}^{(r')}_{\vec{p}'}} \right\} |0\rangle_p$$

$$- \mathbb{1}(t'-t)\; {}_p\langle 0| \left\{ \cancel{} \quad \bar{v}^{(r')}_{\vec{p}'\beta}(\vec{x}',t')\, \hat{d}^{(r')}_{\vec{p}'} \right\}$$

$$\left\{ \cancel{} \quad + v^{(r)}_{\vec{p}\alpha}(\vec{x},t)\, \hat{d}^{(r)+}_{\vec{p}} \right\} |0\rangle_p$$

$${}_p\langle 0| \hat{d}^{(r)}_{\vec{p}}\, \hat{d}^{(r')+}_{\vec{p}'} |0\rangle_p = \delta_{rr'}\, \delta(\vec{p}-\vec{p}')$$

$$= \sum_{r=1}^{2} \int d^3p \Big\{ \mathbb{1}(t-t')\, u^{(r)}_{\vec{p}\alpha}(\vec{x},t)\, \bar{u}^{(r)}_{\vec{p}\beta}(\vec{x}',t')$$

$$- \mathbb{1}(t'-t)\, v^{(r)}_{\vec{p}\alpha}(\vec{x},t)\, \bar{v}^{(r)}_{\vec{p}\beta}(\vec{x}',t') \Big\}$$

propagator valid for all massive particles

$$= \int d^3p \frac{mc^2}{(2\pi\hbar)^3 E_{\vec p}} \Big\{ \Theta(t-t')\, e^{-\frac{i}{\hbar}\left[E_{\vec p}(t-t') - \vec p(\vec x-\vec x')\right]} P^u_{\alpha\beta}(p)$$

$$- \Theta(t'-t)\, e^{+\frac{i}{\hbar}\left[E_{\vec p}(t-t') - \vec p(\vec x-\vec x')\right]} \underbrace{P^v_{\alpha\beta}(p)}_{= - P^u_{\alpha\beta}(-p)} \Big\}$$

polarization sums

$$P^u_{\alpha\beta}(p) = \sum_{r=1}^{2} u^{(r)}_{\vec p\alpha} \bar u^{(r)}_{\vec p\beta} = \sum_{r=1}^{2} \psi^{(r)}_{\vec p\alpha} \bar\psi^{(r)}_{\vec p\beta} = \ldots = \frac{p_\mu\gamma^\mu + mc}{2mc}$$

$$P^v_{\alpha\beta}(p) = \sum_{r=1}^{2} v^{(r)}_{\vec p\alpha} \bar v^{(r)}_{\vec p\beta} = \sum_{r=3}^{4} \psi^{(r)}_{\vec p\alpha} \bar\psi^{(r)}_{\vec p\beta} = \ldots = \frac{p_\mu\gamma^\mu - mc}{2mc}$$

$$P^v(p) = -P^u(-p)$$

$$r=1,2: \quad \psi^{(r)}_{\vec p} = \frac{1}{\sqrt 2}\begin{pmatrix} \sqrt{\frac{p\sigma}{mc}}\,\chi\left(\frac{(-1)^{r+1}}{2}\right) \\ \sqrt{\frac{p\bar\sigma}{mc}}\,\chi\left(\frac{(-1)^{r+1}}{2}\right)\end{pmatrix}$$

$$r=3,4: \quad \psi^{(r)}_{\vec p} = \frac{1}{\sqrt 2}\begin{pmatrix} \sqrt{\frac{p\sigma}{mc}}\,\chi^c\left(\frac{(-1)^{r}}{2}\right) \\ -\sqrt{\frac{p\bar\sigma}{mc}}\,\chi^c\left(\frac{(-1)^{r}}{2}\right)\end{pmatrix}$$

Note: Klein-Gordon recovered for $P^u(p) \equiv 1$

6.22 Four-dimensional Fourier Representation:

$$S_{\alpha\beta}(\vec x, t; \vec x', t') = \int d^3p \frac{mc^2}{(2\pi\hbar)^3 E_{\vec p}}$$

$i\hbar\partial_\mu$

$$\Big\{ \Theta(t-t')\, e^{-\frac{i}{\hbar}\left[E_{\vec p}(t-t') - \vec p(\vec x-\vec x')\right]}\, \frac{p_\mu \gamma^\mu_{\alpha\beta} + mc\,\delta_{\alpha\beta}}{mc}$$

$$+\, \Theta(t'-t)\, e^{+\frac{i}{\hbar}\left[E_{\vec p}(t-t') - \vec p(\vec x-\vec x')\right]}\, \frac{-p_\mu \gamma^\mu_{\alpha\beta} + mc\,\delta_{\alpha\beta}}{mc} \Big\}$$

$$= \frac{i\hbar\,\partial_\mu \gamma^\mu_{\alpha\beta} + mc\,\delta_{\alpha\beta}}{2mc}\; \underbrace{G_{KG}(\vec x, t; \vec x', t')}_{\text{Klein - Gordon propagator}}$$

$$S_D(x;x') = \underbrace{\frac{\overbrace{i\hbar\,\partial_\mu}^{=\hat p_\mu}\gamma^\mu + mc}{2mc}}_{= p^\mu(i\hbar\partial)}\; G_{KG}(x;x')$$

valid for any massive particles

$$\downarrow$$

$$S(x;x') = \lim_{\varepsilon \downarrow 0} \int \frac{d^4p}{(2\pi\hbar)^4}\, \frac{i\hbar}{p_\mu\gamma^\mu - mc + i\varepsilon}\, e^{-\frac{i}{\hbar} p(x-x')}$$

Feynman prescription

Chapter 7: Relativistic Light-Matter Interaction

Introduction:

- QED: all charged fermionic particles, interaction Maxwell theory
- 3 stages of description:

> relativistic mechanics: "minimal coupling" (for all gauge theories)
> first quantization
> second "

- Result: second quantized Hamilton operator of QED
- Treat interaction perturbatively
- Generic scattering problem, described by scattering operator whose matrix elements determine the cross-section