7 Relativistic Light-Matter Interaction

7.1 Relativistic Mechanics:

7.1.1 Basic Principles:

trajectory described by some parameter σ

$$\left(x^\mu(\sigma)\right) = \begin{pmatrix} c\,t(\sigma) \\ \vec{x}(\sigma) \end{pmatrix}, \quad \left(\dot{x}^\mu(\sigma)\right) = \left(\frac{dx^\mu(\sigma)}{d\sigma}\right) = \begin{pmatrix} c\,\frac{dt(\sigma)}{d\sigma} \\ \frac{d\vec{x}(\sigma)}{d\sigma} \end{pmatrix}$$

$$\mathcal{A}[x^\lambda(\cdot)] = \int_{\sigma_i}^{\sigma_f} d\sigma \; L\left(x^\lambda(\sigma), \dot{x}^\lambda(\sigma)\right)$$

Hamilton principle: Euler-Lagrange equations

$$\frac{\delta \mathcal{A}}{\delta x^\lambda(\sigma)} = \frac{\partial L}{\partial x^\lambda(\sigma)} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\lambda(\sigma)} = 0$$

mechanical gauge invariance: regauging of L

$$L'(x^\lambda, \dot{x}^\lambda) = L(x^\lambda, \dot{x}^\lambda) + \underbrace{\frac{d}{d\sigma}\chi(x^\lambda)}_{\boxed{= \partial_\nu \chi(x^\lambda)\,\dot{x}^\nu}}$$

gauge function

$$\mathcal{A}' = \mathcal{A} + \underbrace{\chi(\sigma_f) - \chi(\sigma_i)}_{\text{surface term}}$$

surface term $\rightarrow$ does not affect equation of motion

$$\frac{\partial L'}{\partial x^\mu} - \frac{d}{d\sigma}\frac{\partial L'}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\mu} + \underbrace{\left(\partial_\mu\partial_\nu\chi - \partial_\nu\partial_\mu\chi\right)}_{=0 \text{ (Schwarz theorem)}}\dot{x}^\nu = 0$$

- Lagrange function must be a Lorentz scalar

Also imposed: form invariance of action with respect to the choice of the trajectory parameter $\boxed{\sigma \longrightarrow \sigma(\sigma')}$

guaranteed once the Lagrange function is homogeneous in the four velocity $\dot{x}^\lambda$ of degree one: $\boxed{L(x^\lambda; \alpha\dot{x}^\lambda) = \alpha^1 L(x^\lambda; \dot{x}^\lambda)}$

Proof:

$$A = \int_{\sigma_i}^{\sigma_f} d\sigma\, L\Big(x^\lambda(\sigma); \underbrace{\dot{x}^\lambda(\sigma)}_{=\frac{dx^\lambda(\sigma)}{d\sigma}}\Big) \overset{\sigma=\sigma(\sigma')}{=} \int_{\sigma_i'}^{\sigma_f'} d\sigma' \frac{d\sigma}{d\sigma'} \underbrace{L\Big(\underbrace{x^\lambda(\sigma(\sigma'))}_{=x^\lambda(\sigma')}; \underbrace{\frac{d\sigma'}{d\sigma}}_{=\alpha} \frac{dx^\lambda(\sigma(\sigma'))}{d\sigma'}\Big)}_{=\frac{d\sigma'}{d\sigma} L\left(x^\lambda(\sigma'); \frac{dx^\lambda(\sigma')}{d\sigma'}\right)}$$

$$1 = \frac{d\sigma}{d\sigma'}\frac{d\sigma'}{d\sigma}$$

$$= A'$$

7.1.2 Free Particle:

Minkowski space-time: $g_{\mu\nu}$

distance between two space-time points x^μ and $x^\mu + dx^\mu$

$$ds = \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu} = \sqrt{c^2 dt^2 - d\vec{x}^2} \quad \text{Lorentz invariant}$$

rest frame: $d\vec{x}_R = \vec{0} \implies \underbrace{ds}_{\text{proper length}} = c\, \underbrace{d\tau}_{\text{proper time}}$

Length of a trajectory

$$\int_{s_i}^{s_f} ds = \int_{\sigma_i}^{\sigma_f} d\sigma \, \frac{ds}{d\sigma} = \int_{\sigma_i}^{\sigma_f} d\sigma \underbrace{\sqrt{g_{\mu\nu} \, \dot{x}^\mu(\sigma) \, \dot{x}^\nu(\sigma)}}_{\text{Lorentz invariant and hom. in velocity of degree 1}}$$

$\Rightarrow$ could be a candidate for a Lagrange function

$$A^{(0)} = -Mc \int_{\sigma_i}^{\sigma_f} d\sigma \sqrt{g_{\mu\nu} \, \dot{x}^\mu(\sigma) \dot{x}^\nu(\sigma)}$$

This is the action of a free particle due to correct non-relativistic limit

$$A^{(0)} \underset{\sigma = t}{=} -Mc^2 \int_{t_i}^{t_f} dt \underbrace{\sqrt{1 - \frac{\dot{\vec{x}}^2(t)}{c^2}}}_{\simeq 1 - \frac{1}{2}\frac{\dot{\vec{x}}^2}{c^2}} = \int_{t_i}^{t_f} dt \left\{ -Mc^2 + \frac{M}{2} \dot{\vec{x}}^2(t) + \ldots \right\}$$

$|\dot{\vec{x}}(t)| \ll c$

7.1.2 Charged Particle:

charge q in scalar potential $\varphi(\vec{x}, t)$

$$A^{(int)} = -q \int_{t_i}^{t_f} dt \; \varphi(\vec{x}(t), t) = -q \int_{t_i}^{t_f} dt \; \dot{x}^0(t) \cdot A_0(\vec{x}(t), t)$$

$$\left(\dot{x}^\mu(t)\right) = \begin{pmatrix} c \\ \dot{\vec{x}}(t) \end{pmatrix}, \quad \left(A^\mu(x)\right) = \begin{pmatrix} \varphi/c \\ \vec{A} \end{pmatrix}$$

covariant extension

$$A^{(int)} = -\underbrace{q}_{\text{formal interaction strength}} \int_{\sigma_i}^{\sigma_f} d\sigma \; \overbrace{\dot{x}^\mu(t)}^{\text{four velocity}} \underbrace{A_\mu(x^\lambda(\sigma))}_{\text{maxwell field}}$$

$\Rightarrow$ Lorentz invariant and parametrization invariant

$$\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(int)} \Rightarrow L = -mc\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} - q\,\dot{x}^\mu A_\mu$$

gauge transformation in Maxwell theory $A'_\mu = A_\mu + \partial_\mu \Lambda$

$\Rightarrow$ " " in relat. mechanics: $\chi = -q\Lambda$

Euler-Lagrange equations:

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\mu} = 0 \Rightarrow m\ddot{x}^\mu = m\frac{\ddot{s}}{\dot{s}}\dot{x}^\mu + \frac{\dot{s}}{c} q\, g^{\mu\lambda} \underbrace{F_{\lambda\nu}}_{= \partial_\lambda A_\nu - \partial_\nu A_\lambda} \dot{x}^\nu$$

$$\dot{s}(\sigma) = \sqrt{g_{\mu\nu}\dot{x}^\mu(\sigma)\dot{x}^\nu(\sigma)}$$

$$\underbrace{g^{\mu\lambda} F_{\lambda\nu}}_{F^\mu{}_\nu}$$

special choice of trajectory parameter:

$$\sigma = \tau: \quad s(\tau) = c\tau \Rightarrow \dot{s}(\tau) = c, \quad \ddot{s}(\tau) = 0$$

$$\Rightarrow m\ddot{x}^\mu = q F^\mu{}_\nu \dot{x}^\nu \quad \hat{=} \text{ Lorentz force (relativistic extension)}$$

7.1.3 Minimal coupling:

Action, $\sigma = t$: $\quad \mathcal{A} = \int_{t_i}^{t_f} dt\, L(\vec{x}(t), \dot{\vec{x}}(t), t)$

$$L = -mc^2\sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}} - q\,\varphi(\vec{x},t) + q\,\dot{\vec{x}}(t)\vec{A}(\vec{x},t)$$

Canonical momentum

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = \frac{m\dot{\vec{x}}}{\sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}}} + q\,\vec{A}(\vec{x},t) \Rightarrow \dot{\vec{x}} = \frac{c\,[\vec{p} - q\vec{A}(\vec{x},t)]}{\sqrt{[\vec{p} - \vec{A}(\vec{x},t)q]^2 + m^2c^2}}$$

$$H = \dot{\vec{x}}\,\underbrace{\frac{\partial L}{\partial \dot{\vec{x}}}}_{= \vec{p}_{kin}} - L = \underbrace{c\sqrt{[\vec{p} - q\vec{A}(\vec{x},t)]^2 + m^2c^2}}_{= H_{kin}} + q\varphi(\vec{x},t)$$

$$\overset{c\to\infty}{=} mc^2 + \frac{[\vec{p} - q\vec{A}(\vec{x},t)]^2}{2m} + q\varphi(\vec{x},t) + \cdots$$

Comparison:

free theory $(q=0)$ interacting theory $(q\neq 0)$

$$H_{kin} \longrightarrow H - q\varphi$$

$$\vec{p}_{kin} \longrightarrow \vec{p} - q\vec{A}$$

Covariant formulation:

$$\boxed{\begin{aligned} (p^\mu) = \begin{pmatrix} H/c \\ \vec{p} \end{pmatrix} &\longrightarrow p^\mu - qA^\mu, \quad (A^\mu) = \begin{pmatrix} \varphi/c \\ \vec{A} \end{pmatrix} \\ p_\mu &\longrightarrow p_\mu - qA_\mu \end{aligned}}$$

Minimal coupling

Now: 1st quantization $\rightarrow$ Jordan rule $p_\mu \rightarrow \hat{p}_\mu = i\hbar\partial_\mu$

7.2 QED Action:

Starting point: free Dirac theory

$\hat{p}_\mu = i\hbar\partial_\mu \xrightarrow[\text{coupling}]{\text{minimal}} \hat{p}_\mu - qA_\mu$

$$\mathcal{L} = \bar{\Psi}(x)\left(i\hbar c\gamma^\mu\partial_\mu - mc^2\right)\Psi(x)$$

global U(1) phase transformation invariance

$$\Psi'(x) = e^{-\frac{i}{\hbar}q\underbrace{\boxed{\Lambda}}_{\text{constant}}}\Psi(x), \quad \bar{\Psi}'(x) = e^{+\frac{i}{\hbar}q\Lambda}\bar{\Psi}(x) \Rightarrow \mathcal{L}' = \mathcal{L}$$

Noether: conservation of charge

Local U(1) gauge transformation:

$$\Psi'(x) = e^{-\frac{i}{\hbar}q\Lambda(x)}\Psi(x), \quad \bar{\Psi}'(x) = e^{\frac{i}{\hbar}q\Lambda(x)}\bar{\Psi}(x)$$

Lagrange density is NOT invariant

$$\partial_\mu\Psi'(x) = e^{-\frac{i}{\hbar}q\Lambda(x)}\left\{\partial_\mu\Psi(x) \underbrace{- \frac{iq}{\hbar}\partial_\mu\Lambda(x)\Psi(x)}_{\text{changes } \mathcal{L}}\right\}$$

$$\mathcal{L}' = \mathcal{L} + \underbrace{qc\bar{\Psi}(x)\gamma^\mu\partial_\mu\Lambda(x)\Psi(x)}_{\text{additional term due to local U(1) transformation}}$$

Local gauge invariance established provided a four-vector potential exists

$$\partial_\mu \Psi(x) \xrightarrow[\text{coupling}]{\text{minimal}} D_\mu \Psi(x) = \left(\partial_\mu + \frac{iq}{\hbar} A_\mu(x) \right) \Psi(x)$$

$$\underbrace{D_\mu' \Psi'(x) \stackrel{!}{=} e^{-\frac{i}{\hbar} q \Lambda(x)} \cdot D_\mu \Psi(x)}_{\text{covariant derivative transforms like } \Psi(x)}$$

$$D_\mu^{*\prime} \overline{\Psi}'(x) \stackrel{!}{=} e^{+\frac{i}{\hbar} q \Lambda(x)} D_\mu^{*} \overline{\Psi}(x)$$

$$\mathcal{L} = \overline{\Psi}(x) \left(i\hbar c \gamma^\mu \underbrace{D_\mu}_{\text{with minimal coupling}} - m c^2 \right) \Psi(x)$$

$$= \overline{\Psi}(x) \left[i\hbar c \gamma^\mu \left(\partial_\mu + \frac{iq}{\hbar} A_\mu \right) - m c^2 \right] \Psi(x)$$

not complete yet, add free Maxwell theory

$$\mathcal{L}^{(0)} = \overline{\Psi}(x) \left(i\hbar c \gamma^\mu \partial_\mu - m c^2 \right) \Psi(x) - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L}^{(int)} = \underbrace{- q c \overline{\Psi}(x) \gamma^\mu \Psi(x)}_{= j^\mu(x) \leftarrow \text{current density of Dirac theory}} A_\mu(x) = - j^\mu(x) A_\mu(x)$$

Euler - Lagrange equations:

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu, \quad i\gamma^\mu D_\mu \Psi - \frac{mc}{\hbar} \Psi = 0, \quad i D_\mu^{*} \overline{\Psi} \gamma^\mu + \frac{mc}{\hbar} \overline{\Psi} = 0$$

7.3 QED Hamilton Function:

$$\mathcal{L} = \bar\psi\left(i\hbar c\gamma^\mu\partial_\mu - mc^2\right)\psi + \frac{\varepsilon_0}{2}\left(\frac{\partial\vec A(\vec x,t)}{\partial t}\right)^2 - \frac{1}{2\mu_0}\left[\vec\nabla\times\vec A\right]^2$$
$$+\varepsilon_0\frac{\partial\vec A}{\partial t}\vec\nabla\varphi + \frac{\varepsilon_0}{2}\left(\vec\nabla\varphi\right)^2 - q\,\bar\psi\gamma^0\psi\varphi + q\bar\psi\vec\gamma\psi\vec A$$

Coulomb gauge: $\operatorname{div}\vec A = 0$

$$\varphi(\vec x,t) = \int d^3x'\,\frac{\rho(\vec x',t)}{4\pi\varepsilon_0|\vec x-\vec x'|}\,,\qquad \rho(\vec x,t) = q\,\psi^+(\vec x,t)\psi(\vec x,t)$$

$\Rightarrow$ no radiation gauge, $\varphi \neq 0$

$$\pi = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial t}\right)} = i\hbar\bar\psi\gamma^0 = i\hbar\psi^+\,,\qquad \bar\pi = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\bar\psi}{\partial t}\right)} = 0$$

$$\vec\pi = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\vec A}{\partial t}\right)} = \varepsilon_0\left\{\frac{\partial\vec A}{\partial t} + \vec\nabla\varphi\right\}$$

Legendre transformation:

$$\mathcal{H} = \pi\frac{\partial\psi}{\partial t} + \frac{\partial\bar\psi}{\partial t}\bar\pi + \vec\pi\cdot\frac{\partial\vec A}{\partial t} - \mathcal{L}$$

$$H = \int d^3x\,\mathcal{H} = \int d^3x\left\{\psi^+(\vec x,t)\left(-i\hbar c\vec\alpha\vec\nabla + mc^2\beta\right)\psi(\vec x,t) + \frac{\varepsilon_0}{2}\left(\frac{\partial\vec A(\vec x,t)}{\partial t}\right)^2\right.$$
$$+\frac{1}{2\mu_0}\partial_k A_l(\vec x,t)\,\partial_k\partial A_l(\vec x,t) + \cancel{\frac{\varepsilon_0}{2}\varphi(\vec x,t)\Delta\varphi(\vec x,t)} + \frac{q}{2}\psi^+(\vec x,t)\psi(\vec x,t)\varphi(\vec x,t)$$
$$\left.-qc\,\psi^+\vec\alpha\psi\vec A\right\}$$

$$= \int d^3x'\,\frac{q\,\psi^+(\vec x',t)\psi(\vec x',t)}{|\vec x-\vec x'|}$$

$$H = \underbrace{H^{(0)}}_{\text{Dirac + Maxwell}} + H^{(int)}$$

$$H^{(int)} = -qc \int d^3x\, \overline{\Psi}(\vec{x},t)\, \vec{\gamma}\, \Psi(\vec{x},t)\, \vec{A}(\vec{x},t)$$

$$+ \frac{q^2}{8\pi\varepsilon_0} \int d^3x \int d^3x' \, \frac{\overline{\Psi}(\vec{x},t)\gamma^0\Psi(\vec{x},t)\, \overline{\Psi}(\vec{x}',t)\gamma^0\Psi(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$$

this comes from free Dirac term and minimal coupling $\vec{\nabla} \to \vec{\nabla} - \frac{iq}{\hbar}\vec{A}$

instantaneous Coulomb self interaction

contradicts special relativity, is there due to Coulomb gauge

Note: This term will be cancelled by a non-covariant term in Maxwell propagator once a cross section is calculated