7.4 Dirac Picture:

Introduction:

- Free QFT: "trivial"
  - Hamiltonian quadratic in quantum field operators
  - character of creation/annihilation do not change with time
- Interacting QFT: "non-trivial"
  - nonlinearities lead to interesting physics
  - character of creation/annihilation operator can change with time

    $t=0$: annihilation operator

    $\downarrow$

    $t>0$: e.g. superposition of annihilation/creation particles
  - not exactly solvable
- Weak interaction: perturbation theory
  - Schrödinger picture: states time dependent, operators time-independent NO
  - Heisenberg " : " " independent, " " dependent NO
  - Dirac " : states and operators time dependent

7.4.1 Derivation:

- starting point: $\hat{H}_S = \hat{H}_S^{(0)} + \hat{H}_S^{(int)}$ (closed quantum system) (no explicit time dependence)

Schrödinger free interacting

- Schrödinger picture:

$$i\hbar \frac{\partial}{\partial t} |\Psi_S(t)\rangle = \hat{H}_S |\Psi_S(t)\rangle \Rightarrow |\Psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_S t} |\Psi_S(0)\rangle$$

- Dirac picture:

$$|\Psi_D(t)\rangle = e^{+\frac{i}{\hbar}\hat{H}_S^{(0)} t} |\Psi_S(t)\rangle \Leftrightarrow |\Psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_S^{(0)} t} |\Psi_D(t)\rangle \quad (*)$$

- Dynamics of operators:

no time dependence

$$\langle \Psi_D(t)| \hat{O}_D(t) |\Psi_D(t)\rangle \overset{!}{=} \langle \Psi_S(t)| \hat{O}_S |\Psi_S(t)\rangle$$

$$\overset{(*)}{=} \langle \Psi_D(t)| \boxed{\underbrace{e^{\frac{i}{\hbar}\hat{H}_S^{(0)} t}\, \hat{O}_S\, e^{-\frac{i}{\hbar}\hat{H}_S^{(0)} t}}_{=\hat{O}_D(t)}} |\Psi_D(t)\rangle \quad (**)$$

- Example: $\hat{O}_S = \hat{H}_S^{(0)}$

$$\hat{H}_D^{(0)}(t) = e^{\frac{i}{\hbar}\hat{H}_S^{(0)} t}\, \hat{H}_S^{(0)}\, e^{-\frac{i}{\hbar}\hat{H}_S^{(0)} t} = \hat{H}_S^{(0)}$$

- Equations of motion: operators / states ?

$$i\hbar \frac{\partial}{\partial t} \underline{|\Psi_D(t)\rangle} = e^{\frac{i}{\hbar}\hat{H}_S^{(0)} t} \Big\{ \underbrace{i\hbar \frac{\partial}{\partial t} |\Psi_S(t)\rangle}_{= \left(\cancel{\hat{H}_S^{(0)}} + \hat{H}_S^{(int)}\right) |\Psi_S(t)\rangle} - \cancel{\hat{H}_S^{(0)}} |\Psi_S(t)\rangle \Big\}$$

$$\overset{(*)}{=} \underbrace{e^{\frac{i}{\hbar}\hat{H}_S^{(0)} t}\, \hat{H}_S^{(int)}\, e^{-\frac{i}{\hbar}\hat{H}_S^{(0)} t}}_{= \hat{H}_D^{(int)}(t)} |\Psi_D(t)\rangle \quad (**)$$

Tomonaga-Schwinger equation

$$i\hbar \frac{\partial}{\partial t} \hat{O}_D(t) \overset{(**)}{=} e^{\frac{i}{\hbar}\hat{H}_S^{(0)} t}\left(\hat{O}_S \hat{H}_S^{(0)} - \hat{H}_S^{(0)} \hat{O}_S\right) e^{-\frac{i}{\hbar}\hat{H}_S^{(0)} t}$$

$$= e^{\frac{i}{\hbar}\hat{H}_S^{(0)} t}\, \hat{O}_S\, e^{-\frac{i}{\hbar}\hat{H}_S^{(0)} t}\, \hat{H}_S^{(0)} - \hat{H}_S^{(0)}\, e^{\frac{i}{\hbar}\hat{H}_S^{(0)} t}\, \hat{O}_S\, e^{-\frac{i}{\hbar}\hat{H}_S^{(0)} t}$$

$$= \left[\hat{O}_D(t), \hat{H}_S^{(0)}\right]_-$$

Heisenbergs equation

Why is this advantageous?

Field operators in Dirac picture turn out to retain their respective properties of the free theory to create or annihilate a particle

7.4.2 Example: non-relativistic bosons

- Schrödinger picture $\hat{\Psi}_S^\dagger(\vec{x})$, $\hat{\Psi}_S(\vec{x})$

$$[\hat{\Psi}_S(\vec{x}), \hat{\Psi}_S(\vec{x}')]_- = 0 = [\hat{\Psi}_S^\dagger(\vec{x}), \hat{\Psi}_S^\dagger(\vec{x}')]_- , \quad [\hat{\Psi}_S(\vec{x}), \hat{\Psi}_S^\dagger(\vec{x}')]_- = \delta(\vec{x}-\vec{x}')$$

- basis $u_{\vec{p}}(\vec{x})$

$$\int d^3x\, u^*_{\vec{p}}(\vec{x})\, u_{\vec{p}'}(\vec{x}) = \delta(\vec{p}-\vec{p}') , \quad \int d^3p\, u^*_{\vec{p}}(\vec{x})\, u_{\vec{p}}(\vec{x}') = \delta(\vec{x}-\vec{x}')$$

- expansion

$$\hat{\Psi}(\vec{x}) = \int d^3p\, u_{\vec{p}}(\vec{x})\, \hat{a}_{\vec{p}} \quad \Rightarrow \quad \hat{a}_{\vec{p}} = \int d^3x\, u^*_{\vec{p}}(\vec{x})\, \hat{\Psi}(\vec{x})$$

$$\hat{\psi}^+(\vec{x}) = \int d^3p\, u_{\vec{p}}^*(\vec{x})\, \hat{a}^+_{\vec{p}} \quad \Rightarrow \quad \hat{a}^+_{\vec{p}} = \int d^3x\, u_{\vec{p}}(\vec{x})\, \hat{\psi}^+(\vec{x})$$

$$[\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}'}]_- = 0 = [\hat{a}^+_{\vec{p}}, \hat{a}^+_{\vec{p}'}]_-, \quad [\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{p}'}]_- = \delta(\vec{p}-\vec{p}')$$

- free Hamiltonian in Schrödinger picture

$$\hat{H}_S^{(0)} = \int d^3p\, E_{\vec{p}}\, \underbrace{\hat{a}^+_S(\vec{p})\,\hat{a}_S(\vec{p})}_{=\hat{n}_S(\vec{p})}$$

- Heisenberg equation in Dirac picture

$$i\hbar \frac{\partial \hat{a}_D(\vec{p},t)}{\partial t} = [\hat{a}_D(\vec{p},t), \hat{H}_S^{(0)}]_- = e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t}\, [\hat{a}_S(\vec{p}), \hat{H}_S^{(0)}]_-\, e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t}$$

$$= \int d^3p'\, E_{\vec{p}'}\, e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t}\, \underbrace{[\hat{a}_S(\vec{p}), \hat{a}^+_S(\vec{p}')\hat{a}_S(\vec{p}')]_-}_{= \frac{\delta}{\delta \hat{a}^+_S(\vec{p})}\left\{\hat{a}^+_S(\vec{p}')\hat{a}_S(\vec{p}')\right\} = \hat{a}_S(\vec{p}')\,\delta(\vec{p}-\vec{p}')}\, e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t}$$

$$= E_{\vec{p}}\, \underbrace{e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t}\, \hat{a}_S(\vec{p})\, e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t}}_{=\hat{a}_D(\vec{p},t)} \quad ; \quad \hat{a}_D(\vec{p},0) = \hat{a}_S(\vec{p})$$

$$\Rightarrow \quad \hat{a}_D(\vec{p},t) = \underbrace{e^{-\frac{i}{\hbar}E_{\vec{p}}t}}_{\text{phase factor}}\, \hat{a}_S(\vec{p}), \quad \hat{a}^+_D(t) = e^{\frac{i}{\hbar}E_{\vec{p}}t}\, \hat{a}^+_S(\vec{p})$$

$[\hat{a}_D(\vec{p},t), \hat{a}_D(\vec{p}',t)]_- = 0 = [\hat{a}_D^+(\vec{p},t), \hat{a}_D^+(\vec{p}',t)]_-$, $[\hat{a}_D(\vec{p},t), \hat{a}_D^+(\vec{p}',t)]_- = \delta(\vec{p}-\vec{p}')$

$\Rightarrow$ $\hat{a}_D(\vec{p},t)$, $\hat{a}_D^+(\vec{p},t)$ are annihilation/creation operators of particle with momentum $\vec{p}$ at time t

$\Rightarrow$ $\hat{\Psi}_D(\vec{x},t)$, $\hat{\Psi}_D^+(\vec{x},t)$: in analogy

7.5 Canonical Field Quantization for QED:

Notation: D can be dropped, always Dirac picture

- Dirac field:

$$[\hat{\Psi}_\alpha(\vec{x},t), \hat{\Psi}_\beta(\vec{x}',t)]_+ = [\hat{\pi}_\alpha(\vec{x},t), \hat{\pi}_\beta(\vec{x}',t)]_+ = 0, \quad [\hat{\Psi}_\alpha(\vec{x},t), \hat{\pi}_\beta(\vec{x}',t)]_+ = i\hbar\,\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}')$$

spinor degrees of freedom

- Maxwell field:

$$[\hat{A}_k(\vec{x},t), \hat{A}_l(\vec{x}',t)]_- = 0 = [\hat{\pi}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_-, \quad [\hat{A}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_- = i\hbar\,\delta^T_{kl}(\vec{x}-\vec{x}')$$

vector degrees of freedom

$$= \delta_{kl}\,\delta(\vec{x}-\vec{x}') + \underbrace{\frac{1}{4\pi}\partial_k\partial_l'\frac{1}{|\vec{x}-\vec{x}'|}}_{\text{due to Coulomb gauge}}$$

- Independence of Dirac/Maxwell operators:

$$[\hat{\Psi}_\alpha(\vec{x},t), \hat{A}_k(\vec{x}',t)]_- = [\hat{\Psi}_\alpha(\vec{x},t), \hat{\pi}_k(\vec{x}',t)]_- = 0$$

$$[\hat{\pi}_\alpha(\vec{x},t), \hat{A}_k(\vec{x}',t)]_- = [\hat{\pi}_\alpha(\vec{x},t), \hat{\pi}_k(\vec{x}',t)]_- = 0$$

- Momentum operators:

$$\hat{\pi}(\vec{x},t) = i\hbar\, \hat{\bar{\Psi}}(\vec{x},t)\gamma^0 = i\hbar\, \hat{\Psi}^+(\vec{x},t)$$

$$\hat{\vec{\pi}}(\vec{x},t) = \varepsilon_0 \left\{ \frac{\partial \hat{\vec{A}}(\vec{x},t)}{\partial t} + \underbrace{\vec{\nabla}\hat{\varphi}(\vec{x},t)}_{\text{key problem}} \right\}, \quad \hat{\varphi}(\vec{x},t) = \int d^3x' \, \frac{q\, \hat{\Psi}^+(\vec{x}',t)\hat{\Psi}(\vec{x}',t)}{4\pi\varepsilon_0 |\vec{x}-\vec{x}'|}$$

still Coulomb gauge
but no radiation gauge

- change of point of view:

$$\hat{\pi}_\alpha(\vec{x},t), \quad \hat{\pi}_k(\vec{x},t) \longrightarrow \hat{\Psi}^+_\alpha(\vec{x},t), \quad \frac{\partial \hat{A}_k(\vec{x},t)}{\partial t}$$

- Dirac field:

$$[\hat{\Psi}_\alpha(\vec{x},t), \hat{\Psi}_\beta(\vec{x}',t)]_+ = [\hat{\Psi}^+_\alpha(\vec{x},t), \hat{\Psi}^+_\beta(\vec{x}',t)]_+ = 0, \quad [\hat{\Psi}_\alpha(\vec{x},t), \hat{\Psi}^+_\beta(\vec{x}',t)]_+ = \delta(\vec{x}-\vec{x}')$$

- Dirac and Maxwell field:

$$[\hat{\Psi}_\alpha(\vec{x},t), \hat{A}_k(\vec{x}',t)]_- = 0 = [\hat{\Psi}^+_\alpha(\vec{x},t), \hat{A}_k(\vec{x}',t)]_-$$

$$\left[\hat{\Psi}_\alpha(\vec{x},t), \frac{\partial \hat{A}_k(\vec{x}',t)}{\partial t}\right]_- = \ldots = -\frac{q}{4\pi\varepsilon_0} \frac{(\vec{x}-\vec{x}')_k}{|\vec{x}-\vec{x}'|^3}\hat{\Psi}_\alpha(\vec{x},t)$$

$$\underbrace{[\hat{\Psi}^+_\alpha(\vec{x},t), \quad - \quad - \quad - \quad \text{analog}}_{\text{non-locality due to Coulomb gauge}}$$

- Maxwell field:

$$[\hat{A}_k(\vec{x},t), \hat{A}_l(\vec{x}',t)]_- = 0, \quad \left[\hat{A}_k(\vec{x},t), \frac{\partial \hat{A}_l(\vec{x}',t)}{\partial t}\right]_- = \ldots = \frac{i\hbar}{\varepsilon_0}\bar{\delta}_{kl}(\vec{x}-\vec{x}')$$

$$\left[\frac{\partial \hat{A}_k(\vec{x},t)}{\partial t}, \frac{\partial \hat{A}_l(\vec{x}',t)}{\partial t}\right]_- = \ldots = 0$$

- Interacting Hamilton operator of QED and normal ordering

$$\hat{H}^{(int)} = -qc \int d^3x \, :\hat{\bar{\Psi}}(\vec{x},t)\,\vec{\gamma}\,\hat{\Psi}(\vec{x},t):\,\hat{\vec{A}}(\vec{x},t)$$ ← minimal coupling

$$+\frac{q^2}{8\pi\varepsilon_0}\int d^3x \int d^3x' \, \frac{:\hat{\bar{\Psi}}(\vec{x},t)\gamma^0\hat{\Psi}(\vec{x},t)\,\hat{\bar{\Psi}}(\vec{x}',t)\gamma^0\hat{\Psi}(\vec{x}',t):}{|\vec{x}-\vec{x}'|}$$ ← its effects are cancelled by non-covariant terms in Maxwell propagator

7.6 Time Evolution Operator:

$$i\hbar\frac{\partial}{\partial t}|\Psi_D(t)\rangle = \hat{H}_D^{(int)}(t)\,|\Psi_D(t)\rangle \Rightarrow |\Psi_D(t)\rangle = \underbrace{\hat{U}_D(t_2,t_1)}_{\text{time evolution operator in Dirac picture}}|\Psi_D(t_1)\rangle$$

difficulty

formally solved

Formal solution in Schrödinger picture

$$|\Psi_S(t_2)\rangle = e^{-\frac{i}{\hbar}\hat{H}_S(t_2-t_1)}|\Psi_S(t_1)\rangle$$

$$|\Psi_D(t_2)\rangle = e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}|\Psi_S(t_2)\rangle = \underbrace{e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}\,e^{-\frac{i}{\hbar}\hat{H}_S(t_2-t_1)}\,e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_1}}_{=\hat{U}_D(t_2,t_1)}|\Psi_D(t_1)\rangle$$

Note: $[\hat{H}_S^{(0)},\hat{H}_S]_- \neq 0 \Rightarrow$ operator ordering essential!

- Properties:
  - initial condition: $\hat{U}_D(t_1,t_1) = 1$
  - group property: $\hat{U}_D(t_3,t_2)\,\hat{U}_D(t_2,t_1) = \dots = \hat{U}_D(t_3,t_1)$
  - inverse: $\hat{U}_D(t_1,t_2)\hat{U}_D(t_2,t_1) = \hat{U}_D(t_1,t_1) = 1 \Rightarrow \hat{U}_D^{-1}(t_2,t_1) = \hat{U}_D(t_1,t_2)$

- unitary: $\hat{U}_D^{+}(t_2,t_1) = \hat{U}_D(t_1,t_2) = \hat{U}_D^{-1}(t_2,t_1)$

- differential equation:

$$i\hbar\frac{\partial}{\partial t_2}\hat{U}_D(t_2,t_1) = e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}\underbrace{(-\hat{H}_S^{(0)}+\hat{H}_S}_{=\hat{H}_S^{(int)}})e^{-\frac{i}{\hbar}\hat{H}_S(t_2-t_1)}\,e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_1}$$

$$= 1 = e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}$$

$$= \hat{H}_D^{(int)}(t_2)\,\hat{U}_D(t_2,t_1)$$

$$\hat{U}_D(t_2,t_1) = 1 - \frac{i}{\hbar}\int_{t_1}^{t_2}dt_1'\,\hat{H}_D^{(int)}(t_1')\,\hat{U}_D(t_1',t_1)$$

iterative solution $\rightarrow$ von Neumann series

$$= 1 - \frac{i}{\hbar}\int_{t_1}^{t_2}dt_1'\,\hat{H}_D^{(int)}(t_1) + \left(-\frac{i}{\hbar}\right)^2\int_{t_1}^{t_2}dt_1'\int_{t_1}^{t_1'}dt_2'\,\hat{H}_D^{(int)}(t_1')\,\hat{H}_D^{(int)}(t_2')$$

$+ \ldots$ n order $+ \ldots$

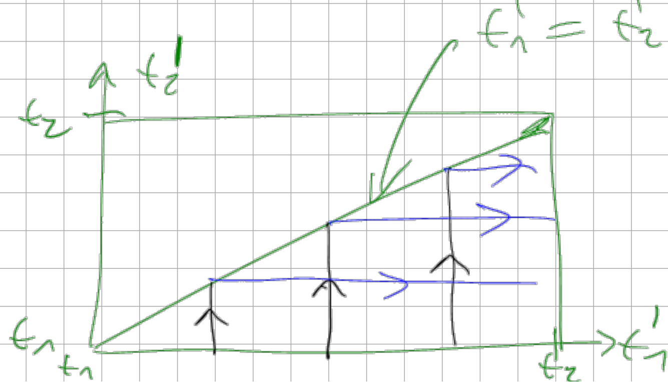
time ordering $t_1' > t_2' > \ldots > t_n'$
in multiple integrals

Freeman Dyson: all integrals can run over full intervall $[t_1,t_2]$

$\Rightarrow$ time ordering operator

$$\int_{t_1}^{t_2}dt_1'\int_{t_1}^{t_1'}dt_2'\,\hat{H}_D^{(int)}(t_1')\,\hat{H}_D^{(int)}(t_2')$$

$$= \int_{t_1}^{t_2}dt_2'\int_{t_2'}^{t_2}dt_1'\,\hat{H}_D^{(int)}(t_1')\,\hat{H}_D^{(int)}(t_2')$$



$$\overset{t_1' \leftrightarrow t_2'}{=} \int_{t_1}^{t_2} dt_1' \int_{t_1'}^{t_2} dt_2' \, \hat{H}_D^{(int)}(t_2') \, \hat{H}_D^{(int)}(t_1')$$

$$= \frac{1}{2} \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_2} dt_2' \left\{ \Theta(t_1' - t_2') \, \hat{H}_D^{(int)}(t_1') \, \hat{H}_D^{(int)}(t_2') + \Theta(t_2' - t_1') \, \hat{H}_D^{(int)}(t_2') \, \hat{H}_D^{(int)}(t_1') \right\}$$

$$= \hat{T}\left(\hat{H}_D^{(int)}(t_1') \, \hat{H}_D^{(int)}(t_2') \right)$$

generalise to order n

$$\hat{U}(t_2, t_1) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar}\right)^n \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_2} dt_2' \cdots \int_{t_1}^{t_2} dt_n' \, \hat{T}\left(\hat{H}^{(int)}(t_1') \, \hat{H}^{(int)}(t_2') \cdots \hat{H}^{(int)}(t_n') \right)$$

$$= \hat{T} \exp\left\{ -\frac{i}{\hbar} \int_{t_1}^{t_2} dt_1' \, \hat{H}^{(int)}(t_1') \right\}$$

7.7 Scattering Operator:

$$|\psi(t)\rangle = \hat{U}(t, -\infty) \underbrace{|\psi_i\rangle}_{= |\psi(-\infty)\rangle}$$

scattering matrix:

transition amplitude at time t to be in $|\psi_f\rangle$

$$S_{fi} = \lim_{t \to \infty} \langle \psi_f | \psi(t) \rangle = \lim_{t \to \infty} \langle \psi_f | \hat{U}(t, -\infty) | \psi_i \rangle$$

$$= \langle \psi_f | \underbrace{\hat{S}}_{= \hat{U}(+\infty, -\infty)} | \psi_i \rangle$$

$$= \hat{T} \exp\left\{ -\frac{i}{\hbar} \int_{-\infty}^{+\infty} dt\, H^{(int)}(t) \right\}$$

$\hat{S}$ in QED:

$$\hat{S} = \hat{T} \exp\left\{ \frac{iq}{\hbar} \int d^4x \; :\hat{\bar{\Psi}}(x) \vec{\gamma}\, \hat{\Psi}(x): \hat{\vec{A}}(x) \right.$$

$$\left. - \frac{iq^2}{8\pi\hbar\varepsilon_0} \int_{-\infty}^{+\infty} dt \int d^3x \int d^3x' \frac{:\hat{\bar{\Psi}}(\vec{x},t)\,\gamma^0\, \hat{\Psi}(\vec{x},t)\, \hat{\bar{\Psi}}(\vec{x}',t)\,\gamma^0\, \hat{\Psi}(\vec{x}',t)}{|\vec{x}-\vec{x}'|} \right\}$$

Expansion formal in q