Chapter 8 Møller Scattering

Motivation:

- Cross section of elastic scattering of 2 electrons: $e^- + e^- \to e^- + e^-$
  named after Danish physicist Christian Møller
- Møller scattering:
  - fundamental scattering process in QED
  - dominant in low energy (< 100 MeV) electron scattering events
  - deviations from it at higher energies allows to check standard model and to study possibly physics beyond standard model
- Outline:
  - Scattering matrix up to second in $q = -e$ $(e > 0)$: cancellation of non-covariant terms, covariant result can be graphically represented by Feynman diagrams
  - Unknown polarization of initial and final electrons allow to average the square of scattering matrix with respect to the polarization → Clifford algebra of Dirac matrix
  - Kinematics of relativistic two-particle scattering is analysed with Mandelstam variables: in center of mass reference frame the Mandelstam variables turn out to be related to scattering energies and to scattering angle

• Scattering cross section for Møller scattering: ultra-relativistic and non-relativistic limit (Rutherford cross section in forward peak)

8.1 Scattering matrix:

$S_{fi} = \langle \Psi_f | \hat{S} | \Psi_i \rangle$, $\hat{S} = \hat{U}(+\infty, -\infty)$

$$\hat{S} \overset{QED}{=} 1 + \frac{iq}{\hbar} \int d^4x : \hat{\bar{\Psi}}(x) \vec{\gamma}\, \hat{\Psi}(x) : \hat{\vec{A}}(x)$$

Taylor expansion in $q = -e$

$$+ \frac{1}{2}\left(\frac{+iq}{\hbar}\right)^2 \int d^4x \int d^4x' \hat{T}\left\{\left[: \hat{\bar{\Psi}}(x) \vec{\gamma}\, \hat{\Psi}(x) : \hat{\vec{A}}(x)\right]\left[: \hat{\bar{\Psi}}(x') \vec{\gamma}\, \hat{\Psi}(x') : \hat{\vec{A}}(x')\right]\right\}$$

$$- \frac{iq^2}{8\pi\hbar\varepsilon_0} \int dt \int d^3x \int d^3x' \frac{: \hat{\bar{\Psi}}(\vec{x},t)\gamma^0 \hat{\Psi}(\vec{x},t)\, \hat{\bar{\Psi}}(\vec{x}',t)\gamma^0 \hat{\Psi}(\vec{x}',t) :}{|\vec{x} - \vec{x}'|} + \dots$$

initial scattering state: $|\Psi_i\rangle = |\vec{p}_{i1}, s_{i1}; \vec{p}_{i2}, s_{i2}\rangle$

final " " : $|\Psi_f\rangle = |\vec{p}_{f1}, s_{f1}; \vec{p}_{f2}, s_{f2}\rangle$

orthogonal: $\langle \Psi_i | \Psi_f \rangle = 0$ $\hat{=}$ different moment $\Rightarrow$ no contribution in zeroth order $S^{(0)}_{fi}$

also $S^{(1)}_{fi} = 0$ as $|\Psi_i\rangle, |\Psi_f\rangle$ do not contain a photon

Lowest order of scattering matrix: second order in $q = -e$

$$\Rightarrow \quad S^{(2)}_{fi} = S^{(2,inst)}_{fi} + S^{(2,rad)}_{fi}$$

$$S^{(2,inst)}_{fi} = \frac{-ie^2}{8\pi\hbar\varepsilon_0 c^2}\int dt \int d^3x \int d^3x' \frac{\langle\Psi_f| :\hat{j}^0(\vec{x},t)\,\hat{j}^0(\vec{x}',t): |\Psi_i\rangle}{|\vec{x}-\vec{x}'|}$$

$$S^{(2,rad)}_{fi} = -\frac{e^2}{2\hbar^2c^2}\int d^4x \int d^4x' \langle\Psi_f|\,T\left[:\hat{j}^k(x)\hat{A}_k(x): \; :\hat{j}^l(x')\hat{A}_l(x'):\right]|\Psi_i\rangle$$

$$\hat{j}^\mu(x) = c\,\hat{\bar{\Psi}}(x)\,\gamma^\mu\,\hat{\Psi}(x)$$

$$\hat{\Psi}(x) = \int d^3p_2 \sum_{s_2}\sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\vec{p}_2}}}\left\{e^{\frac{i}{\hbar}p_2x}\,\bar{u}(\vec{p}_2,s_2)\,\hat{b}^+_{\vec{p}_2,s_2} + e^{-\frac{i}{\hbar}p_2x}\,\bar{v}(\vec{p}_2,s_2)\,\hat{d}_{\vec{p}_2,s_2}\right\}$$

$\hat{\Psi}^+(x) =$ analogue $\hat{b}_{\vec{p}_1,s_1}$ $\hat{d}^+_{\vec{p}_1,s_1}$

$$\hat{j}^\mu(x) = c\int d^3p_1\int d^3p_2 \sum_{s_1}\sum_{s_2}\sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\vec{p}_1}}}\sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\vec{p}_2}}}$$

$$\Big\{e^{\frac{i}{\hbar}(p_2-p_1)x}\;\bar{u}(\vec{p}_1,s_2)\,\gamma^\mu u(\vec{p}_2,s_2)\;\hat{b}^+_{\vec{p}_2,s_2}\,\hat{b}_{\vec{p}_1,s_1}$$

$$+ (*)\;\cancel{\hat{b}^+_{\vec{p}_2,s_2}\,\hat{d}^+_{\vec{p}_1,s_1}} + (*)\;\cancel{\hat{d}_{\vec{p}_2,s_2}\,\hat{b}_{\vec{p}_1,s_1}} + (*)\;\cancel{\hat{d}_{\vec{p}_2,s_2}\,\hat{d}^+_{\vec{p}_1,\vec{p}_2}}\Big\}$$

$$S^{(2,inst)}_{fi} = \frac{-ie^2}{8\pi\hbar\varepsilon_0c^2}\int dt\int d^3x\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\int d^3p_1\int d^3p_2\int d^3p_3\int d^3p_4 \sum_{s_1}\sum_{s_2}\sum_{s_3}\sum_{s_4}$$

$$\sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\vec{p}_1}}}\sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\vec{p}_2}}}\sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\vec{p}_3}}}\sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\vec{p}_4}}}$$

$$e^{\frac{i}{\hbar}(E_{\vec{p}_2}-E_{\vec{p}_1})t-\frac{i}{\hbar}(\vec{p}_2-\vec{p}_1)\vec{x}}\quad e^{\frac{i}{\hbar}(E_{\vec{p}_4}-E_{\vec{p}_3})t-\frac{i}{\hbar}(\vec{p}_4-\vec{p}_4)\vec{x}'}$$

$$\bar{u}(\vec{p}_2,s_2)\,\gamma^0 u(\vec{p}_1,s_1)\;\bar{u}(\vec{p}_4,s_4)\,\gamma^0 u(\vec{p}_3,s_3))$$

$$\bullet\ C(\vec p_1, \vec p_2, \vec p_3, \vec p_4; s_1, s_2, s_3, s_4)$$

$$\langle 0 | \underbrace{\hat b_{\vec p_{f2}, s_{f2}}\, \hat b_{\vec p_{f1}, s_{f1}} : \hat b^+_{\vec p_2 s_2}}\ \underbrace{\hat b_{\vec p_1 s_1}\ \hat b^+_{\vec p_4, s_4}\ \hat b_{\vec p_3 s_3} :}\ \hat b^+_{\vec p_{i1}, s_1}\ \hat b^+_{\vec p_{i2}, s_{i2}} | 0 \rangle$$

$$= - \hat b^+_{\vec p_2, s_2}\, \hat b^+_{\vec p_4 s_4}\, \hat b_{\vec p_1 s_1}\, \hat b_{\vec p_3 s_3}$$

$$\frac{1}{|\vec x - \vec x'|} = \int \frac{d^3 k}{(2\pi)^3} \frac{4\pi}{k^2} e^{i \vec k (\vec x - \vec x')}$$

$$S^{(2, inst)}_{fi} = \frac{-i \hbar e^2}{2 \varepsilon_0 c} \boxed{} \boxed{(2\pi\hbar)^4 \delta^{(4)}(p_2 + p_4 - p_1 - p_3)}$$

energy – momentum conservation

$$\frac{1}{(\vec p_2 - \vec p_1)^2} \bar u(\vec p_2, s_2) \gamma^0 u(\vec p_1, s_1)\, \bar u(\vec p_4, s_4) \gamma^0 u(\vec p_3, s_3)\, \underline{C(\vec p_1, \vec p_2, \ldots s_3, s_4)}$$

$$S^{(2, rad)}_{fi} = - \frac{e^2}{2 \hbar^2 c^2} \int d^4 x \int d^4 x' \{ \Theta(x^0 - x'^0) \langle \Psi_f | j^{\mu}(x) A_{\mu}(x) j^{\nu}(x') A_{\nu}(x') | \Psi_i \rangle$$

$$+ \Theta(x'^0 - x^0) \langle \Psi_f | j^{\nu}(x') A_{\nu}(x') j^{\mu}(x) A_{\mu}(x) | \Psi_i \rangle \}$$

$\hat A_0(x) \equiv 0$ radiation gauge in unperturbed theory

$$= - \frac{e^2}{2\hbar^2 c^2} \int d^4 x \int d^4 x' \{ \Theta(x^0 - x'^0) \langle 0 | A_\mu(x) A_\nu(x') | 0 \rangle$$

$$\langle \Psi_f | :j^\mu(x): :j^\nu(x')| \Psi_i \rangle + \Theta(x'^0 - x^0) \langle 0 | A_\nu(x') A_\mu(x) | 0 \rangle$$

$$\langle \Psi_f | :j^\nu(x'): :j^\mu(x) | \Psi_i \rangle \}$$

$$\parallel = c^2 \boxed{} e^{\frac{i}{\hbar}(p_2 - p_1)x} \; e^{\frac{i}{\hbar}(p_4 - p_3)x'}$$

$$\bar{u}(\vec{p}_2, s_2)\gamma^\mu u(\vec{p}_1, s_1)\; \bar{u}(\vec{p}_4, s_4)\gamma^\nu u(\vec{p}_3, s_3)$$

$$\langle 0 | \hat{b}_{\vec{p}_{f2}, s_{f2}} \hat{b}_{\vec{p}_{f1}, s_{f1}} \hat{b}^+_{\vec{p}_2 s_2} \underbrace{\hat{b}_{\vec{p}_1 s_1} \hat{b}^+_{\vec{p}_4, s_4}} \hat{b}_{\vec{p}_3, s_3} \hat{b}^+_{\vec{p}_{i1}, s_{i1}} \hat{b}^+_{\vec{p}_{i2}, s_{i2}} | 0 \rangle$$

$$= -\hat{b}^+_{\vec{p}_4 s_4} \hat{b}_{\vec{p}_1 s_1} + \cancel{\delta(\vec{p}_1 - \vec{p}_4)\delta_{s_1, s_4}}$$

$$= C(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4; s_1, s_2, s_3, s_4)$$

= can be evaluated, leads to 4 terms

(*)

$$= C(\vec{p}_3, \vec{p}_4; \vec{p}_1, \vec{p}_2; s_4, s_3, s_1, s_2)$$

$$\vec{p}_1, s_1 \longleftrightarrow \vec{p}_3, s_3 \; ; \quad \vec{p}_2, s_2 \longleftrightarrow \vec{p}_4, s_4$$

$$S^{(2,\text{rad})}_{fi} = -\frac{e^2}{2\hbar^2} \int d^4x \int d^4x' \boxed{} D_{\mu\nu}(x, x')$$

$$e^{\frac{i}{\hbar}(p_2 - p_1)x} \; e^{\frac{i}{\hbar}(p_4 - p_3)x'} \; \bar{u}(\vec{p}_2, s_2)\gamma^\mu u(\vec{p}_1, s_1)\, \bar{u}(\vec{p}_4, s_4)\gamma^\nu u(\vec{p}_3, s_3)$$

$$\cdot\, \underline{C(\vec{p}_1, \vec{p}_2, \_\_ \_\_ \_\_ \_\_)}$$

$$D_{\mu\nu}(x, x') = \frac{i\hbar}{c\varepsilon_0} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} e^{ik(x-x')} P_{\mu\nu}(k)$$

$$S^{(2,\,\mathrm{rad})}_{fi} = \frac{-i e^2 \hbar}{2\varepsilon_0 c} \boxed{\qquad} (2\pi\hbar)^4 \delta(p_2+p_4-p_1-p_3)$$

$$\frac{P_{\mu\nu}(p_2-p_1)}{(p_2-p_1)^2}\, \underbrace{\bar u(p_2,s_2)\gamma^\mu u(p_1,s_1)}\, \underbrace{\bar u(p_4,s_4)\gamma^\nu u(p_3,s_3)}\; C(\ -\ -\ -\ )$$

$$P_{\mu\nu}(k) = \underbrace{-g_{\mu\nu}}_{\text{covariant}} - k^2 \frac{\xi_\mu \xi_\nu}{(k\xi)^2-k^2} - \underbrace{\cancel{\frac{k_\mu k_\nu + (k\xi)(k_\mu \xi_\nu + k_\nu \xi_\mu)}{(k\xi)^2-k^2}}}_{\text{(vanishes due to charge conservation) Ward–Takahashi identity}}$$

→ Feynman propagator

non-covariant: $(\xi^\lambda) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ time-like vector

Proof:

$$\bar u(\vec p_2,s_2)\gamma^\mu u(\vec p_1,s_1)(p_{2\mu}-p_{1\mu}) \overset{\text{Dirac equation}}{=} -mc\,\bar u(\vec p_2,s_2)u(\vec p_1,s_1) + mc\,\bar u(\vec p_2,s_2)u(\vec p_1,s_1) = 0$$

$$= \frac{1}{(\vec p_2-\vec p_1)^2}$$

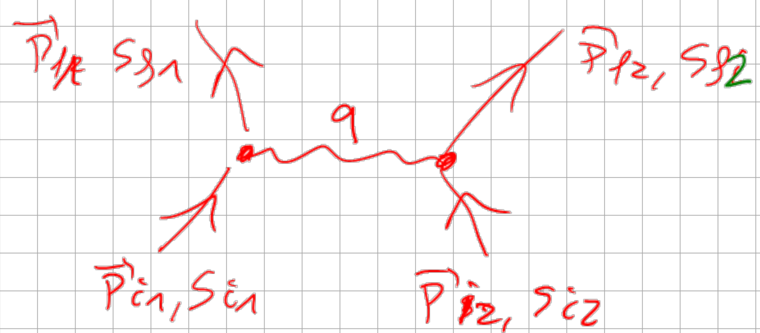
Cancellation of non-covariant terms!

$$S^{(2)}_{fi} = \frac{i e^2 \hbar^2}{\varepsilon_0 c}(2\pi\hbar)^4 \delta(p_{f1}+p_{f2}-p_{i2}-p_{i1}) \sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{i1}}}}\sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{i2}}}}\sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{f1}}}}\sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{f2}}}}$$

$$\Big\{ \frac{g_{\mu\nu}}{(p_{f1}-p_{i1})^2}\, \bar u(\vec p_{f1},s_{f1})\gamma^\mu u(\vec p_{i1},s_{i1})\, \bar u(\vec p_{f2},s_{f2})\gamma^\nu u(\vec p_{i2},s_{i2})$$

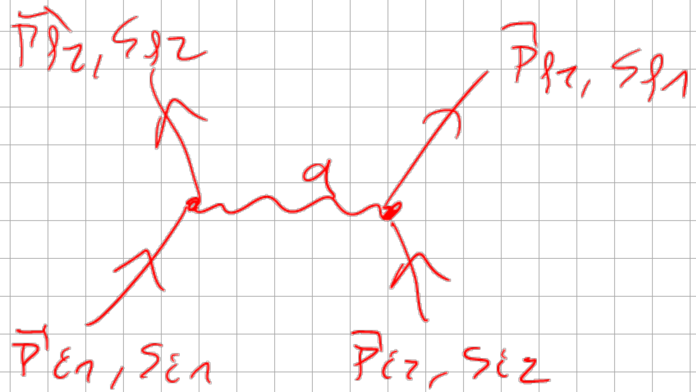
$$- \frac{g_{\mu\nu}}{(p_{f1}-p_{i2})}\, \bar u(\vec p_{f1},s_{f1})\gamma^\mu u(\vec p_{i2},s_{i2})\, \bar u(\vec p_{f2},s_{f2})\gamma^\nu u(\vec p_{i1},s_{i1}) \Big\}$$

$\hat{=}$



$q = p_{f1} - p_{i1} = p_{f2} - p_{i2}$

direct graph



$q = p_{f2} - p_{i1} = p_{f1} - p_{i2}$

indirect graph

Note: zeroth loop order $\hat{=}$ no internal integral / sum

$\hat{=}$ tree-level diagram