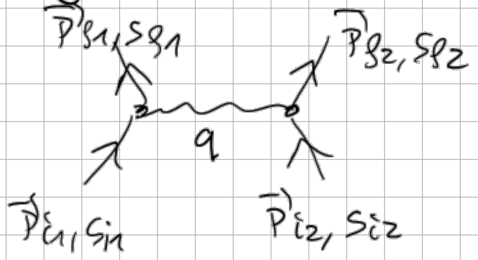
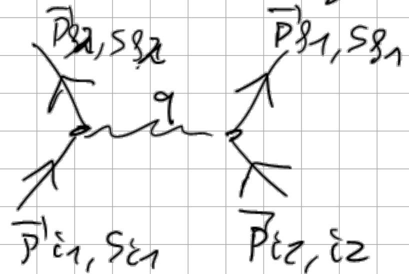
Scattering matrix for Møller scattering in lowest order:

$S^{(2)}_{fi} =$ 

$-$ 

$$= \frac{i\hbar e^2}{\varepsilon_0 c}(2\pi\hbar)^4\, \delta^{(4)}(p_{f1}+p_{f2}-p_{i1}-p_{i2})\sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{i1}}}}\sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{i2}}}}\sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{f1}}}}\sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{f2}}}}\; M^{(2)}_{fi}$$

matrix element involves spinorial amplitudes $u, \bar u$

Absolute square value:

$$|S^{(2)}_{fi}|^2 = \frac{\hbar^2 e^4}{\varepsilon_0^2 c^2}\boxed{(2\pi\hbar)^4\, \delta(0)}\,\delta(p_{f1}+p_{f2}-p_{i1}-p_{i2})\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{i1}}}\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{i2}}}\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{f1}}}\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p_{f2}}}\,|M^{(2)}_{fi}|^2$$

formal infinity

Solution: Reconsider the whole scattering problem

Instead of using plane waves in continuum

$$\hat\Psi(x) = \int d^3p \sum_s \sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p}}}\left\{e^{-\frac{i}{\hbar}px}\, u(\vec p,s)\,\hat b_{\vec p,s} + e^{\frac{i}{\hbar}px}\, v(\vec p,s)\,\hat d^{\dagger}_{\vec p,s}\right\}$$

consider plane wave for a box of volume $V = L^3$ with periodic boundary conditions

$$\hat{\psi}(x) = \sum_{\vec{p}} \sum_s \sqrt{\frac{mc^2}{V E_{\vec{p}}}} \left\{ \quad " \quad \right\}$$

discrete momenta following from periodic conditions

Continuum — box of volume V and finite observational time T

$$\int d^4x \; e^{-\frac{i}{\hbar}(p-p')x} = (2\pi\hbar)^4 \delta(p-p') \Leftrightarrow \int_V d^3x \int_{-\frac{Tc}{2}}^{Tc/2} dx^0 \; e^{-\frac{i}{\hbar}(p-p')x} = VTc \; \delta_{p,p'}$$

$$\boxed{(2\pi\hbar)^4 \delta(0) \quad \Leftrightarrow \quad VTc}$$

$$(2\pi\hbar)^3 \quad \Leftrightarrow \quad V$$

Rules to go from continuum to the box and vice versa

Result in a box:

$$\frac{|S_{fi}^{(2)}|^2}{VT} = \frac{\hbar^2 e^4}{\varepsilon_0^2 c} (2\pi\hbar)^4 \delta(p_{f1} + p_{f2} - p_{i1} - p_{i2}) \frac{(mc^2)^4}{V^4 E_{\vec{p}_{i1}} E_{\vec{p}_{i2}} E_{\vec{p}_{f1}} E_{\vec{p}_{f2}}} |M_{fi}^{(2)}|^2$$

transition rate per volume

- ~~Integrate~~ over final momenta, sum over final helicities (unpolarized)

$$\underbrace{\sum_{\vec{p}_{f1}}} \quad \underbrace{\sum_{\vec{p}_{f2}}} \quad \sum_{s_{f1}} \sum_{s_{f2}}$$

$$\frac{V}{(2\pi\hbar)^3}\int d^3 p_{f1} \quad \frac{V}{(2\pi\hbar)^3}\int d^3 p_{f2}$$

- Average over polarisations of incoming electrons

$$\frac{1}{4} \sum_{s_{i1}} \sum_{s_{i2}}$$

Averaged transition rate per volume:

$$W = \frac{1}{4} \sum_{s_{i1}} \sum_{s_{i2}} \sum_{s_{f1}} \sum_{s_{f2}} \frac{V^2}{(2\pi\hbar)^6} \int d^3 p_{f1} \int d^3 p_{f2} \quad \frac{|S_{fi}^{(2)}|^2}{VT}$$

$$= \frac{e^4 M^2 c^4}{4\pi\varepsilon_0^2 c V^2 E_{\vec{p}_{i1}} E_{\vec{p}_{i2}}} \underbrace{\int d^3 p_{f1} \int d^3 p_{f2}}_{\text{double momentum integral}} \delta^{(4)}(p_{f1} + p_{f2} - p_{i1} - p_{i2}) \frac{M^2 c^4}{E_{\vec{p}_{f1}} E_{\vec{p}_{f2}}} \overline{|M_{fi}^{(2)}|^2}$$

$$= \frac{1}{4} \sum_{s_{i1}} \sum_{s_{i2}} \sum_{s_{f1}} \sum_{s_{f2}} |M_{fi}^{(2)}|^2$$

$\longrightarrow$ last lecture

$$I = \int \frac{d^3 p_{f1}}{2E_{\vec{p}_{f1}}} \int \frac{d^3 p_{f2}}{2E_{\vec{p}_{f2}}} \delta^{(4)}(p_{f1} + p_{f2} - p_{i1} - p_{i2}) \; f(\vec{p}_{f1}, \vec{p}_{f2})$$

auxiliary calculation:

$$\int_0^\infty dp^0 \, \delta^{(1)}(p^2 - M^2c^2) = \int_0^\infty dp^0 \, \delta^{(1)}\Big(\underbrace{(p^0)^2 - \vec{p}^{\,2} - M^2c^2}_{= g(p^0)}\Big)$$

distributional rule:

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \delta(x - x_i) \; ; \quad g(x_i) = 0$$

$$= \ldots = \frac{c}{2E_{\vec p'}}, \qquad E_{\vec p'} = \sqrt{\vec p'^2 c^2 + m^2 c^4}$$

$$I = \frac{1}{c}\int \frac{d^3 p_{f1}}{(2\pi\hbar)^3} \underbrace{\int d^3 p_{f2} \underbrace{\int_0^\infty dp^0_{f2}}_{= \int_{-\infty}^{+\infty} dp^0_{f2}\, \Theta(p^0_{f2})}}_{= \int d^4 p_{f2}} \delta^{(1)}(p_{f2}^2 - m^2c^2)\, \delta^{(4)}(p_{f1} + p_{f2} - p_{i1} - p_{i2})\, g(\vec p_{f1}, \vec p_{f2})$$

$\Rightarrow$ formally evaluate $\delta^{(4)}$-integral:

$$I = \frac{1}{c}\int \frac{d^3 p_{f1}}{(2\pi\hbar)^3}\, \Theta(p^0_{i1} + p^0_{i2} - p^0_{f1})\, \delta^{(1)}\left((p_{i1} + p_{i2} - p_{f1})^2 - m^2c^2\right) g(\vec p_{f1}, \vec p_{i1} + \vec p_{i2} - \vec p_{f1})$$

↑ later on this $\delta^{(1)}$-function imposes $E_{\vec p'} = E_{\vec p}$

center-of-mass reference frame:

$$p_{i1} = \begin{pmatrix} E_{\vec p}/c \\ \vec p \end{pmatrix}, \quad p_{i2} \begin{pmatrix} E_{\vec p}/c \\ -\vec p \end{pmatrix}, \quad p_{f1} = \begin{pmatrix} E_{\vec p'} \\ \vec p' \end{pmatrix}, \quad p_{f2} = \begin{pmatrix} E_{\vec p'} \\ -\vec p' \end{pmatrix}$$

$$p^0_{i1} + p^0_{i2} - p^0_{f1} = \frac{2E_{\vec p} - E_{\vec p'}}{c}, \qquad \vec p_{i1} + \vec p_{i2} - \vec p_{f1} = -\vec p'$$

$$\left[p_{i1} + p_{i2} = \begin{pmatrix} \frac{2E_{\vec p}}{c} \\ \vec 0 \end{pmatrix} \right]$$

$$(p_{i1} + p_{i2} - p_{f1})^2 = (p_{i1} + p_{i2})^2 - 2p_{f1}(p_{i1} + p_{i2}) + p_{f1}^2$$

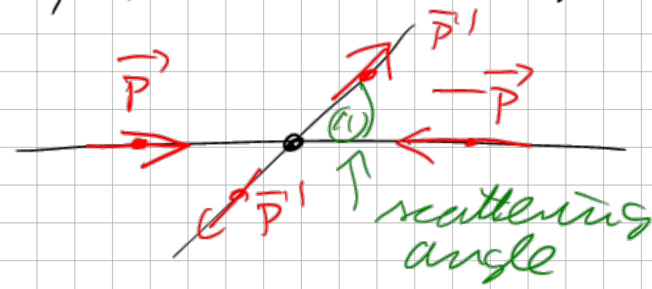
$$= \left(\frac{2E_{\vec p}}{c}\right)^2 - 2\,\frac{2E_{\vec p}}{c} \cdot \frac{E_{\vec p'}}{c} + \left(\frac{E_{\vec p'}}{c}\right)^2 - \vec p'^2 \underset{\uparrow}{=} \frac{4E_{\vec p}(E_{\vec p} - E_{\vec p'})}{c^2} + m^2c^2$$

energy-momentum dispersion

$$I = \frac{1}{c}\int \frac{d^3 p'}{2E_{\vec p'}}\, \underbrace{\cancel{\Theta(2E_{\vec p} - E_{\vec p'})}}_{= 1}\, \underbrace{\delta^{(1)}\left(\frac{4E_{\vec p}(E_{\vec p} - E_{\vec p'})}{c^2}\right)}_{= \frac{c^2}{4E_{\vec p}} \delta^{(1)}(E_{\vec p} - E_{\vec p'})} g(\vec p', -\vec p')$$

$\longrightarrow E_{\vec p} = E_{\vec p'}$

spherical coordinates: $d^3p' = |\vec{p}'|^2 \, d|\vec{p}'| \; d\Omega \, , \; d\Omega = \sin\theta \; d\theta \; d\phi$



$$\overline{|M^{(2)}_{fi}|^2} \hat{=} f(\vec{p}', -\vec{p}') \underset{\substack{\uparrow \\ \text{Mandelstam variables}}}{=} F(|\vec{p}'|, \theta)$$

$$I = \frac{c}{8 E_{\vec{p}}} \int_0^\infty d|\vec{p}'| \, \frac{|\vec{p}'|^2}{E_{\vec{p}'}} \int d\Omega \;\; F(|\vec{p}'|, \theta)$$

$$E_{\vec{p}'} = \sqrt{\vec{p}'^2 c^2 + m^2 c^4} \implies d|\vec{p}'| = \frac{E_{\vec{p}'}}{|\vec{p}'| c^2} \, dE_{\vec{p}'}$$

$$\implies I = \frac{1}{8 c^2 E_{\vec{p}}} \sqrt{E_{\vec{p}}^2 - m^2 c^4} \int d\Omega \;\; F\left(\sqrt{\frac{E_{\vec{p}}^2}{c^2} - m^2 c^2}, \theta\right)$$

$$W = \frac{e^4}{\pi^2 \varepsilon_0^2 c} \, \frac{m^4 c^8}{V^2 E^2} \, \frac{\sqrt{E^2 - m^2 c^4}}{8 c^2 E} \int d\Omega \;\; \overline{|M^{(2)}_{fi}|^2} \quad , \; E = E_{\vec{p}}$$

$$\overline{|M^{(2)}_{fi}|^2_d} + \overline{|M^{(2)}_{fi}|_e} + \overline{|M^{(2)}_{fi}|^2_i}$$

- Mandelstam variables
- center-of-mass reference frame

$$= \ldots = \frac{4(2E^2 - m^2c^4)^2 - (8E^4 - 4m^2c^4E^2 - m^4c^8)\sin^2\theta + (E^2 - m^2c^4)^2 \sin^4\theta}{4 \, m^4 c^4 (E^2 - m^2 c^4)^2 \sin^4\theta}$$

Note: $[W] = 1/s\,m^3$

Cross section needs number of incoming electrons per time and volume

$$:j^\mu(x): = \underbrace{\hat{b}^+ \hat{b}} + \cancel{\hat{b}^+ \hat{d}} + \cancel{\hat{d} \hat{b}} - \cancel{\hat{d}^+ \hat{d}}$$

$$\underline{\langle \Psi_i | :\hat{j}^\mu(x): | \Psi_i \rangle} = c \sum_{\vec{p}_1} \sum_{\vec{p}_2} \sum_{s_1} \sum_{s_2} \sqrt{\frac{mc^2}{V E_{\vec{p}_1}}} \sqrt{\frac{mc^2}{V E_{\vec{p}_2}}}$$

$$e^{\frac{i}{\hbar}(p_2 - p_1)x}\, \bar{u}(\vec{p}_2, s_2)\, \gamma^\mu\, u(\vec{p}_1, s_1) \cdot \underbrace{C(\vec{p}_1, \vec{p}_2; s_1, s_2)}$$

$$= \langle 0 | \hat{b}_{\vec{p}_{i2}, s_{i2}}\, \hat{b}_{\vec{p}_{i1}, s_{i1}}\, \hat{b}^+_{\vec{p}_2, s_2}\, \hat{b}_{\vec{p}_1, s_1}\, \hat{b}^+_{\vec{p}_{i1}, s_{i1}}\, \hat{b}^+_{\vec{p}_{i2}, s_{i2}} | 0 \rangle$$

$$= \dots = \delta_{\vec{p}_1, \vec{p}_2}\, \delta_{s_1 s_2} \left(\delta_{\vec{p}_1, \vec{p}_{i1}}\, \delta_{s_1, s_{i1}} + \delta_{\vec{p}_1, \vec{p}_{i2}}\, \delta_{s_1, s_{i2}} \right)$$

$$\rightarrow = c \frac{mc^2}{V E_{\vec{p}_{i1}}}\, \bar{u}(\vec{p}_{i1}, s_{i1})\, \gamma^\mu\, u(\vec{p}_{i1}, s_{i1}) + (i_1 \rightarrow i_2)$$

average over helicities:

$$j^\mu = \frac{1}{4} \sum_{s_{i1}} \sum_{i2} \langle \Psi_i | :\hat{j}^\mu(x): | \Psi_i \rangle = \frac{p_{i1}^\mu c^2}{V E_{\vec{p}_{i1}}} + \frac{p_{i2}^\mu c^2}{V E_{\vec{p}_{i2}}}$$

center-of-mass reference frame:

current density $\quad j^i = 0$

relative current density: $\Delta j^i = \dfrac{2 p^i c^2}{V E_{\vec{p}}}$, $\quad [\Delta j^i] = \dfrac{1}{s\, m^2}$

Cross section:

$$\sigma = \frac{\text{number of scattered electrons / time volume}}{(\text{number of incoming electrons / time area}) / \text{volume}} = \frac{W}{|\Delta \vec{j}| / V}$$

$$\sigma = \int d\Omega \, \underbrace{\frac{d\sigma}{d\Omega}}_{\text{differential cross section}} , \quad \frac{d\sigma}{d\Omega} = \frac{e^4 m^4 c^4}{16\pi^2 \varepsilon_0^2 E^2} \overline{|M_{fi}^{(2)}|^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{\hbar^2 c^2 \alpha^2}{4E^2} \left\{ 1 - \frac{8E^4 - 4m^2c^4E^2 - m^4c^8}{(E^2 - m^2c^4)^2} \frac{1}{\sin^2\Theta} + \frac{4(2E^2 - m^2c^4)^2}{(E^2 - m^2c^4)^2} \frac{1}{\sin^4\Theta} \right\}$$

Sommerfeld fine structure constant $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137}$

Compton wave length: $\lambda_c = \frac{2\pi\hbar}{mc}$

Bohr radius: $a_B = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$

$$\alpha = \frac{\lambda_c}{2\pi a_B}$$

differential cross section: Møller scattering:

- 1932, Christian Møller: in ultrarelativistic limit, guesses, heuristics
- later: derived by Bethe / Fermi based on QED
- indistinguishability of electrons: forward – backward symmetry

$$\frac{d\sigma(\Theta)}{d\Omega} = \frac{d\sigma(\pi - \Theta)}{d\Omega}$$

- experimental check (1954): 7 %
- ultrarelativistic limit: $E \gg mc^2$ ($mc^2 \overset{\text{electr.}}{=} 0.5$ MeV)

$$\left.\frac{d\sigma}{d\Omega}\right|_{ur}(\Theta) = \frac{\alpha^2\hbar^2c^2}{8E^2} \left\{ \frac{1+\cos^4\Theta/2}{\sin^4\frac{\Theta}{2}} + \frac{2}{\sin^2\frac{\Theta}{2}\cos^2\frac{\Theta}{2}} + \frac{1-\sin^4\frac{\Theta}{2}}{\cos^4\frac{\Theta}{2}} \right\}$$

- Non-relativistic limit: $E = mc^2 + \underbrace{\varepsilon}_{= p^2/2m}$, $\varepsilon \ll mc^2$

$$\left.\frac{d\sigma}{d\Omega}\right|_{nr}(\Theta) = \frac{\alpha^2 \hbar^2 c^2 m^2}{16\, \vec{p}^{\,4}} \left\{ \frac{1}{\sin^4\frac{\Theta}{2}} + \frac{1}{\cos^4\frac{\Theta}{2}} - \frac{1}{\sin^2\frac{\Theta}{2}\cos^2\frac{\Theta}{2}} \right\}$$

$m \to \frac{m}{2}$ $\qquad$ $Z = 1$

$$\left.\frac{d\sigma}{d\Omega}\right|_{R}(\Theta) = \frac{\alpha^2 \hbar^2 c^2 m^2 Z}{4\, \vec{p}^{\,4}} \cdot \frac{1}{\sin^4\frac{\Theta}{2}}$$