2.2 Defining Representation of Lorentz Group

last time: invariance
$$\underbrace{x^\mu x^\nu g_{\mu\nu}}_{\text{one inertial system}} = \underbrace{x'^\mu x'^\nu g_{\mu\nu}}_{\text{another inertial system}} \; ; \quad (x^\mu) = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}, \quad (x_\mu) = \begin{pmatrix} ct \\ -\vec{x} \end{pmatrix}$$

identical

linear transformation $x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$

insert this in invariance:
$$x^\mu x^\nu g_{\mu\nu} = (\Lambda^\sigma{}_\mu \, x^\mu)(\Lambda^\varkappa{}_\nu \, x^\nu)\, g_{\sigma\varkappa} = \Lambda^\sigma{}_\mu \Lambda^\varkappa{}_\nu \, g_{\sigma\varkappa} \, x^\mu x^\nu$$

valid for all x^μ:
$$g_{\mu\nu} = \underbrace{\Lambda^\sigma{}_\mu}_{=(\Lambda^T)_\mu{}^\sigma} \underbrace{g_{\sigma\varkappa}}_{\text{minkowski metric}} \Lambda^\varkappa{}_\nu \Rightarrow \boxed{g = \Lambda^T g \Lambda} \quad (*)$$

defining relation for Lorentz transformations Λ

transposed matrix: $(\Lambda^T)_\mu{}^\sigma = g_{\mu\varkappa} (\Lambda^T)^{\varkappa\sigma} = g_{\mu\varkappa} \Lambda^{\sigma\varkappa} = \Lambda^\sigma{}_\mu$

$(*)$: different interpretation $\underbrace{g^\mu{}_\nu}_{=\delta^\mu{}_\nu} = (\Lambda^T)^\mu{}_\sigma \underbrace{g^\sigma{}_\varkappa}_{=\delta^\sigma{}_\varkappa} \Lambda^\varkappa{}_\nu$, $I = (\delta^\mu{}_\nu) \Rightarrow I = \Lambda^T \Lambda \Rightarrow \boxed{\Lambda^T = \Lambda^{-1}}$

Group axioms fulfilled $\Rightarrow$ Lorentz group $\mathcal{L}$

(A1) Closedness: $\Lambda_1, \Lambda_2 \in \mathcal{L}$

$$(\Lambda_1 \Lambda_2)^T g (\Lambda_1 \Lambda_2) = \Lambda_2^T \underbrace{(\Lambda_1^T g \Lambda_1)}_{= g} \Lambda_2 = g \Rightarrow \Lambda_1 \Lambda_2 \in \mathcal{L}$$

(A2) Associativity: $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{L}$

$$[(\Lambda_1 \Lambda_2) \Lambda_3]^T g [(\Lambda_1 \Lambda_2) \Lambda_3] = \Lambda_3^T (\Lambda_1 \Lambda_2)^T g (\Lambda_1 \Lambda_2) \Lambda_3$$

$$= \Lambda_3^T \underbrace{\Lambda_2^T \underbrace{\Lambda_1^T g \Lambda_1}_{= g} \Lambda_2}_{= g} \Lambda_3 = g = [\Lambda_1 (\Lambda_2 \Lambda_3)]^T g \Lambda_1 (\Lambda_2 \Lambda_3)$$

$$\Rightarrow (\Lambda_1 \Lambda_2) \Lambda_3 = \Lambda_1 (\Lambda_2 \Lambda_3)$$

(A3) Identity element: $\Lambda_e = I = (g^\mu{}_\nu) = (\delta^\mu{}_\nu)$

$\Lambda_e^T g \Lambda_e = g \Rightarrow \Lambda_e \in \mathcal{L}$; $\Lambda \in \mathcal{L} \Rightarrow \Lambda_e \Lambda = \Lambda \Lambda_e = \Lambda$

(A4) Inverse element: $\Lambda \in \mathcal{L} \Rightarrow g = \Lambda^T g \Lambda$

$\mathrm{Det}\, g = \mathrm{Det}(\Lambda^T g \Lambda) = (\mathrm{Det}\, \Lambda^T)\, \mathrm{Det}\, g\, (\mathrm{Det}\, \Lambda^T) \Rightarrow (\mathrm{Det}\, \Lambda)^2 = 1$ $\Big\}$ $\mathrm{Det}\, \Lambda \neq 0$, Λ^{-1} exists $\Rightarrow \Lambda^{-1} = \Lambda^T$

set of all Lorentz transformations = pseudo-orthogonal group

$O(\underbrace{1,3}_{\text{entries of Minkowskian metric}})$

$\Rightarrow$ classified due to 2 properties:

- determinant: $(\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1$

  $\det \Lambda = +1$: special Lorentz transformation

  $\det \Lambda = -1$: non-special " "

- specialization $(*)$ for $\mu = \nu = 0$

  $1 = g_{00} = \Lambda^{\sigma}{}_{0}\, g_{\sigma\alpha}\, \Lambda^{\alpha}{}_{0} = (\Lambda^{0}{}_{0})^2 - \sum_{i=1}^{3} (\Lambda^{i}{}_{0})^2 \Rightarrow (\Lambda^{0}{}_{0})^2 = 1 + \sum_{i=1}^{3} (\Lambda^{i}{}_{0})^2 \geq 1$

  $\Lambda^{0}{}_{0} \geq 1$: orthochronous Lorentz transformation

  $\Lambda^{0}{}_{0} \leq -1$: non- " "

$\Rightarrow$ 4 branches of Lorentz group

branch	det Λ	$\Lambda^0{}_0$	example
$\mathcal{L}_1$	$+1$	>0	identity: diag $(1, 1, 1, 1)$
$\mathcal{L}_2$	-1	>0	space inversion: diag $(1, -1, -1, -1)$
$\mathcal{L}_3$	-1	<0	time inversion: diag $(-1, 1, 1, 1)$
$\mathcal{L}_4$	$+1$	<0	space-time inversion: diag $(-1, -1, -1, -1)$

Lorentz group is not connected: Lorentz transformation from one branch can not continuously be transformed into a Lorentz-transformation of another branch.

In the following: consider $\mathcal{L}_1$ (special, orthochronous Lorentz transformations $\equiv$ Lorentz transformations for brevity)

2.3 Defining Representation of Lorentz Algebra:

$$\left.\begin{array}{ll} 4\times 4 \text{ matrices} & \hat{=}\ 4\cdot 4 = 16 \text{ degrees of freedom} \\ g = \Lambda^T g \Lambda & \hat{=}\ \frac{1}{2}\cdot 4\cdot 5 = 10 \text{ degrees of freedom} \end{array}\right\} \text{dimension of Lorentz group: } 16 - 10 = 6$$

infinitesimal deviation from unity element: $\Lambda^\mu{}_\nu = \underbrace{g^\mu{}_\nu}_{\text{unity element in } \mathcal{L}_1} + \underbrace{\underbrace{\omega^\mu{}_\nu}_{\text{tiny}}}_{= \text{element of Lorentz algebra}}$ (***)

$g = \Lambda^T g \Lambda \Rightarrow \Lambda^\sigma{}_\mu \Lambda^\rho{}_\nu g_{\sigma\rho} = g_{\mu\nu}$ (*)

(***) in (*) + "index gymnastics" => expand in first of $\omega^\mu{}_\nu$

Result: $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$ (see exercises)

There are $\underline{\underline{6}}$ degrees of freedom for an antisymmetric 4x4 matrix

Lorentz algebra has as many degrees of freedom as Lorentz group

Representation of an element $\omega^\mu{}_\nu$ of Lorentz algebra:

$$\underbrace{\omega^\mu{}_\nu}_{\text{general element of Lorentz algebra}} = g^{\alpha\mu} g^\beta{}_\nu \omega_{\alpha\beta} = \frac{1}{2}\left(g^{\alpha\mu} g^\beta{}_\nu - g^{\beta\mu} g^\alpha{}_\nu\right)\omega_{\alpha\beta} = -\frac{i}{2} \underbrace{(L^{\alpha\beta})^\mu{}_\nu}_{= i\left(g^{\alpha\mu}g^\beta{}_\nu - g^{\beta\mu}g^\alpha{}_\nu\right)\text{ basis elements of Lorentz algebra}} \underbrace{\omega_{\alpha\beta}}_{\text{expansion coefficient}}$$

α, β: characterize basis elements of Lorentz algebra

μ, ν: components of basis elements of Lorentz algebra anti-symmetry with respect to α, β and μ, ν

$$(L^{\alpha\beta})^{\mu}{}_{\nu} = -(L^{\beta\alpha})^{\mu}{}_{\nu} = -(L^{\alpha\beta})_{\nu}{}^{\mu} \quad \text{(see exercises)}$$

commutator between two basis elements

$$[L^{\alpha\beta}, L^{\gamma\delta}]_- = i \underbrace{C^{\alpha\beta\gamma\delta}_{\varepsilon\zeta}}_{\text{structure constants of Lorentz algebra}} L^{\varepsilon\zeta} \quad \text{(see exercises)}$$

$$= g^{\alpha\delta} g^{\beta}{}_{\varepsilon} g^{\gamma}{}_{\zeta} + g^{\beta\gamma} g^{\alpha}{}_{\varepsilon} g^{\delta}{}_{\zeta}$$
$$- g^{\alpha\gamma} g^{\beta}{}_{\varepsilon} g^{\delta}{}_{\zeta} - g^{\beta\delta} g^{\alpha}{}_{\varepsilon} g^{\gamma}{}_{\zeta}$$

$\Uparrow$

Irrespective of the representation of Lorentz algebra $\Rightarrow$ one and the same structure constants of Lorentz algebra appear

W. Greiner, Volume 5 (group theory, Lie groups/algebras)

2.4 Classification of Basis Elements:

basis elements $L^{\alpha\beta}$: sorted in 2 classes

α, β: spatial indices

$$L_k = \frac{1}{2}\underbrace{\varepsilon_{klm}}_{\text{Levi-Civita tensor}} L^{lm}$$

$\varepsilon_{123} = 1$, $\varepsilon_{klm} = -\varepsilon_{lkm} = -\varepsilon_{mlk} = -\varepsilon_{kml}$

$$L_1 = L^{23} = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

L_2, L_3

α, β: spatial-temporal indices

$$M_k = L^{0k}$$

$$M_1 = L^{01} = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

M_2, M_3

$[L_k, L_l]_- = i\varepsilon_{klm}L_m$ $\hat{=}$ subalgebra of Lorentz algebra

$[L_k, M_l]_- = i\varepsilon_{klm}M_m$, $[M_k, M_l]_- = -i\varepsilon_{klm}L_m$ (see exercises)

2.5 Lie Theorem:

Lorentz group in vicinity of unity element = Lorentz algebra

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu - \frac{i}{2}(L^{\alpha\beta})^\mu{}_\nu\, \omega_{\alpha\beta}$$

Converse: evaluating exponential function with elements of Lorentz algebra yields Lorentz group

$$\Lambda = e^{-\frac{i}{2} L^{\alpha\beta}\omega_{\alpha\beta}}$$

$$\frac{1}{2} L^{\alpha\beta}\omega_{\alpha\beta} = \frac{1}{2} \underbrace{L^{ij}}_{=\varepsilon_{ijk}L_k} \omega_{ij} + \underbrace{\frac{1}{2}\Big(L^{0i}\omega_{0i} + \underbrace{L^{i0}}_{=-L^{0i}}\underbrace{\omega_{i0}}_{=\omega_{0i}}\Big)}_{=\underbrace{L^{0i}}_{M_i}\underbrace{\omega_{0i}}_{=\zeta^i}}$$

$$= \underbrace{L_k \cdot \underbrace{\frac{1}{2}\varepsilon_{kij}\omega_{ij}}_{=\varphi^k}}_{=\vec{L}\cdot\vec{\varphi}} + \vec{M}\cdot\vec{\zeta} = \underbrace{\vec{L}}_{\text{vector of rotation generators}} \cdot \underbrace{\vec{\varphi}}_{\text{vector of rotation angles}} + \underbrace{\vec{M}}_{\text{vector of boost generators}} \cdot \underbrace{\vec{\zeta}}_{\text{rapidity}}$$

Summary: $\Lambda = e^{-i(\vec{L}\cdot\vec{\varphi} + \vec{K}\cdot\vec{\zeta})}$

$\vec{\varphi} = \vec{0}$: $\Lambda = e^{-i\vec{K}\cdot\vec{\zeta}}$ boosts

$\vec{\zeta} = \vec{0}$: $\Lambda = e^{-i\vec{L}\cdot\vec{\varphi}}$ rotations