2.6 Rotations:

$$R(\vec{\varphi}) \underset{\text{theorem}}{\overset{\text{Lie}}{=\!=\!=}} e^{-i\vec{L}\cdot\underbrace{\vec{\varphi}}_{\text{vector of rotation angles}}} =$$

evaluation: 1) Taylor expansion
2) Cayley-Hamilton theorem

$$L_1 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad L_2 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad L_3 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Ad 2) $g(\underbrace{A}_{n\times n \text{ matrix}} - \lambda E) =$ characteristic polynomial $= \sum_{j=1}^{n} (-1)^{j+1} \lambda^j a_j = 0$

$$\Rightarrow \sum_{j=1}^{n} (-1)^{j+1} A^j a_j = 0$$

to be solved in problem sheet 2:

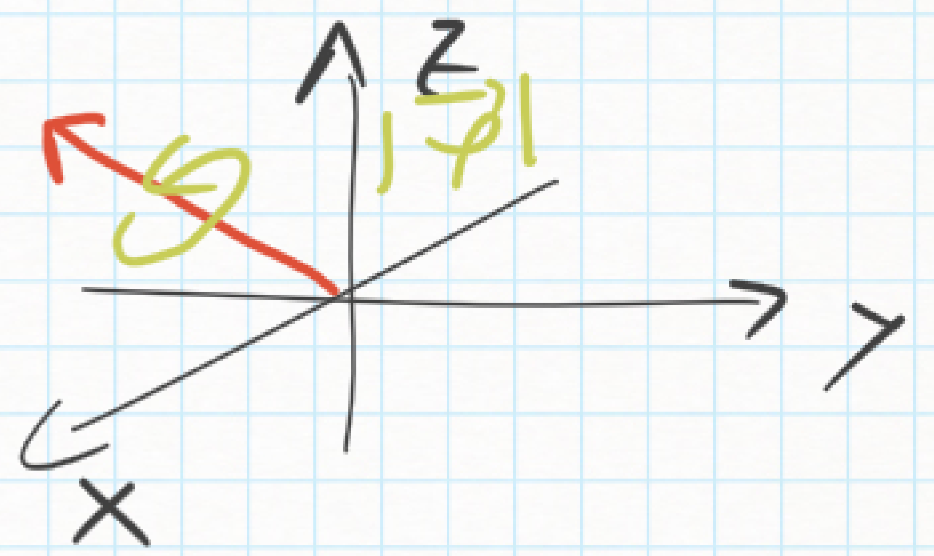
$$R(\vec{\varphi}) = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R_{ij}(\vec{\varphi}) & \\ 0 & & & \end{array}\right)$$

$$R_{ij}(\vec{\varphi}) = \frac{\varphi_i}{|\vec{\varphi}|}\varepsilon_{ijk}\sin|\vec{\varphi}| + \frac{\varphi_j\varphi_k}{|\vec{\varphi}|^2}(1-\cos|\vec{\varphi}|) + \delta_{jk}\cos|\vec{\varphi}|; \quad i,j = 1,2,3$$

fulfills properties, which are characteristic for a rotation:

1) $R(\vec{\varphi})(\vec{\varphi}) = (\vec{\varphi})$, rotation axis $\frac{\vec{\varphi}}{|\vec{\varphi}|}$ is eigenvector of $R(\vec{\varphi})$ with respect to eigenvalue 1

$\frac{\vec{\varphi}}{|\vec{\varphi}|}$



2) $\underbrace{\mathrm{Tr}}_{\text{trace}}\, R(\vec{\varphi}) = 2 + 2\cos|\vec{\varphi}|$

special case: $\vec{\varphi} = \varphi\, \vec{e}_z$

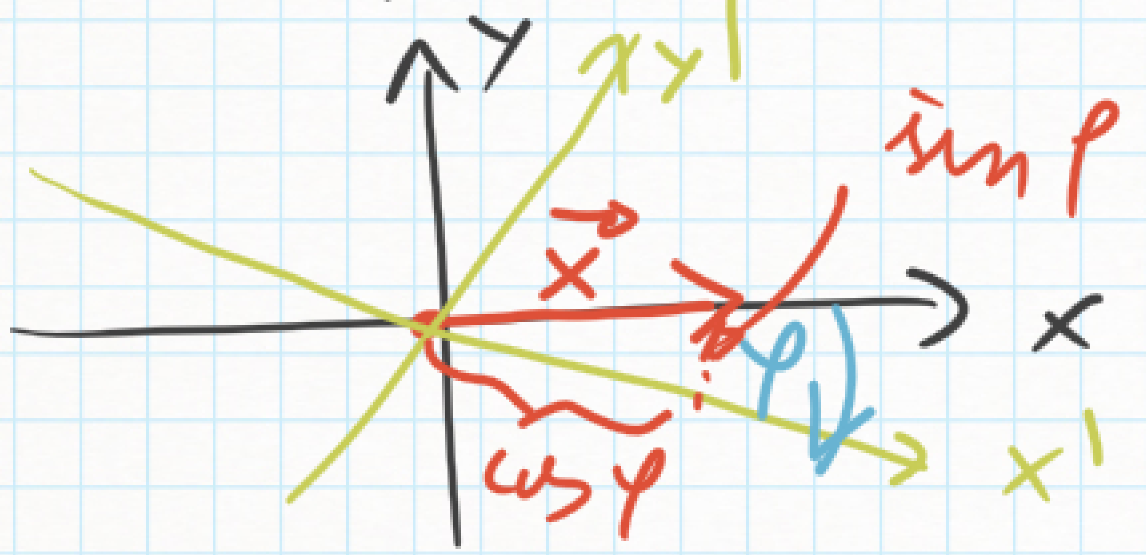
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix};$$

example: $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \vec{x}' = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$

Two different interpretations of a rotation possible:

passive rotation

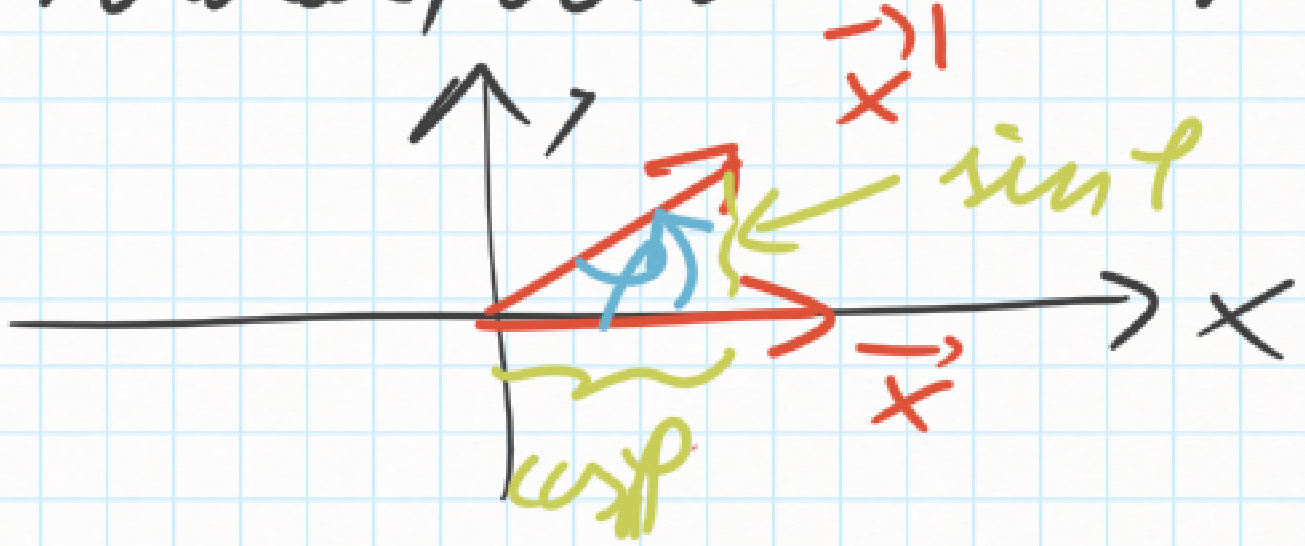
vector fixed, coordinate system rotated



φ: clockwise

active rotation

vector is rotated, coordinate system is fixed



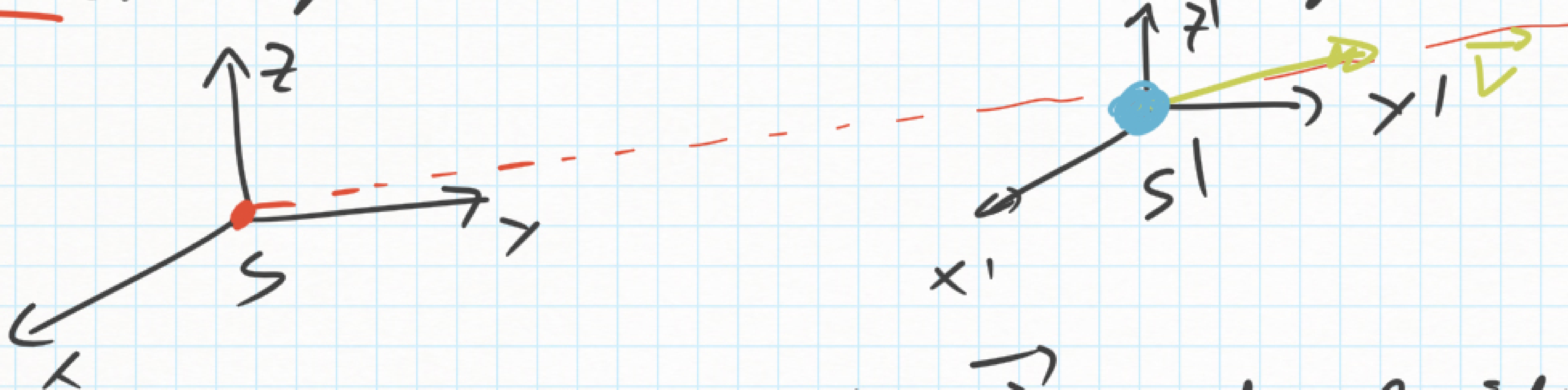
φ anti-clockwise

2.7 Boosts:

$$B(\vec{\xi}) = e^{-i\vec{M}\underbrace{\vec{\xi}}_{\text{rapidities}}} \overset{\substack{\text{1) Taylor expansion} \\ \text{2) Cayley-Hamilton theorem}}}{=} \left(\begin{array}{c|c} \cosh|\vec{\xi}| & \frac{\xi_j}{|\vec{\xi}|}\sinh|\vec{\xi}| \\ \hline \frac{\xi_i}{|\vec{\xi}|}\sinh|\vec{\xi}| & \delta_{ij} + \frac{\xi_i \xi_j}{|\vec{\xi}|^2}\left(\cosh|\vec{\xi}| - 1\right) \end{array}\right)$$

$$M_1 = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_2 = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_3 = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

passive interpretation: boost inertial system S to the inertial system S'



relationship between rapidity $\vec{\xi}$ and velocity $\vec{v}$

description of origin of S' from the point of view of both inertial systems

$$(x^\mu) = \begin{pmatrix} ct \\ \vec{v}t \end{pmatrix}, \quad (x'^\mu) = \begin{pmatrix} ct' \\ \vec{0} \end{pmatrix} : \text{mapping} \quad x'^\mu = B^\mu{}_\nu(\vec{\xi})\, x^\nu$$

$$\mu = 0: \quad t' = t\cosh|\vec{\zeta}| + \frac{\vec{\zeta}}{|\vec{\zeta}|}\cdot\frac{\vec{v}t}{c}\sinh|\vec{\zeta}| \qquad (1)$$

$$\mu = i: \quad \vec{0} = \frac{\vec{\zeta}}{|\vec{\zeta}|}\sinh|\vec{\zeta}| + \frac{\vec{v}}{c} + \frac{\vec{\zeta}\cdot\vec{v}}{|\vec{\zeta}|\cdot c}\frac{\vec{\zeta}}{|\vec{\zeta}|}\left(\cosh|\vec{\zeta}| - 1\right) \qquad (2)$$

solve (2): $$\frac{\vec{\zeta}}{|\vec{\zeta}|} = -\frac{\vec{v}}{|\vec{v}|} \qquad (3)$$ solution ansatz

(3) in (2): $$\vec{0} = -\frac{\vec{v}}{|\vec{v}|}\sinh|\vec{\zeta}| + \cancel{\frac{\vec{v}}{c}} + \frac{\cancel{\vec{v}\cdot\vec{v}}}{\cancel{|\vec{v}|}\,c}\frac{\vec{v}}{\cancel{|\vec{v}|}}\left(\cosh|\vec{\zeta}| \cancel{-1}\right)$$

$$\Rightarrow \quad \frac{|\vec{v}|}{c} = \frac{\sinh|\vec{\zeta}|}{\cosh|\vec{\zeta}|} = \tanh|\vec{\zeta}|$$

$$\cosh|\vec{\zeta}| = \frac{1}{\sqrt{1-\tanh^2|\vec{\zeta}|}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$ Lorentz factor (appears often as abbreviation in special relativity)

$$\sinh|\vec{\zeta}| = \frac{\tanh|\vec{\zeta}|}{\sqrt{1-\tanh^2|\vec{\zeta}|}} = \gamma\frac{|\vec{v}|}{c}$$

$\Rightarrow$ Boost matrix is expressible in terms of the boost velocity representation of a boost:

$$B(\vec{v}) = \left(\begin{array}{c|c} \gamma & -\frac{v_j}{c}\gamma \\ \hline -\frac{v_i}{c}\gamma & \delta_{ij} + \frac{v_i v_j}{|\vec{v}|^2}(\gamma - 1) \end{array}\right)$$

most general boost transformation

conclusion from (1):

$$t' = t\left\{\gamma - \frac{\vec{v}}{\cancel{|\vec{v}|}} \cdot \frac{\vec{v}}{c}\gamma \frac{\cancel{|\vec{v}|}}{c}\right\} = t\,\gamma\left(1 - \frac{\vec{v}^2}{c^2}\right) = \frac{t}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}\left(1 - \frac{\vec{v}^2}{c^2}\right) = t\sqrt{1 - \frac{\vec{v}^2}{c^2}}$$

$\Rightarrow$ time dilation of special relativity

observer in S detects that a clock in S' is moving slower than the clock in S

2.8 Scalar Field Representation:

scalar scalar field $\phi(x^\mu)$: tensor of rank $0 = n$ as it does not change with respect to a Lorentz transformation

passive interpretation:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \Leftrightarrow \quad x^\mu = \underbrace{(\Lambda^{-1})^\mu{}_\nu}_{= \Lambda^T \text{ (see last lecture)}} x'^\nu$$

$\hookrightarrow$ (x^μ) and (x'^μ) describe one and the same space-time point, but once in S and once in S'

invariance: $$\phi(\underbrace{x^\mu}_{= (\Lambda^{-1})^\mu{}_\nu x'^\nu}) = \phi'(x'^\mu)$$

$\Longrightarrow$ omit ' for simplicity

$$\phi'(\cancel{x'}^\mu) = \phi(\underbrace{(\Lambda^{-1})^\mu{}_\nu}_{= g^\mu{}_\nu + \frac{i}{2}(L^{\alpha\beta})^\mu{}_\nu \omega_{\alpha\beta} + \dots} \cancel{x'}^\nu) \quad ; \quad \Lambda^\mu{}_\nu = g^\mu{}_\nu - \frac{i}{2}(L^{\alpha\beta})^\mu{}_\nu \omega_{\alpha\beta} + \dots$$

$$\phi'(x^\mu) = \phi\left(x^\mu + \frac{i}{2}(L^{\alpha\beta})^\mu{}_\nu\, \omega_{\alpha\beta}\, x^\nu + \dots\right)$$

expansion

$$\phi(x^\mu) + \frac{i}{2}(L^{\alpha\beta})^\mu{}_\nu\, \omega_{\alpha\beta}\, x^\nu\, \partial_\mu \phi(\boxed{x^\mu}) + \dots$$

$$= \left(1 - \frac{i}{2}\,\omega_{\alpha\beta}\, \underbrace{\hat{L}^{\alpha\beta}}_{= -(L^{\alpha\beta})^\mu{}_\nu\, x^\nu\, \partial_{\boxed{\mu}}} + \dots\right)\phi(x^\mu)$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

differential operator $\hat{=}$ representation of Lorentz algebra in space of scalar fields

$$(L^{\alpha\beta})^\mu{}_\nu = i\left(g^{\alpha\mu} g^\beta{}_\nu - g^{\beta\mu} g^\alpha{}_\nu\right)$$

$$\Rightarrow \hat{L}^{\alpha\beta} = -i\left(g^{\alpha\mu} g^\beta{}_\nu - g^{\beta\mu} g^\alpha{}_\nu\right) x^\nu\, \partial_\mu = -i\left(x^\beta \partial^\alpha - x^\alpha \partial^\beta\right)$$

four-momentum operator

$$\hat{p}^\alpha = i\hbar\,\partial^\alpha, \quad \hat{L}^{\alpha\beta} = \underbrace{\frac{1}{\hbar}\left(x^\alpha \hat{p}^\beta - x^\beta \hat{p}^\alpha\right)}_{\text{dimensionless orbital angular momentum operator}}$$

$$[\hat{p}^\alpha, x^\beta]_- = i\hbar[\partial^\alpha, x^\beta]_- = i\hbar\,\partial^\alpha x^\beta = i\hbar\, g^{\alpha\mu} \underbrace{\partial_\mu x^\beta}_{= \frac{\partial}{\partial x^\mu} x^\beta = \delta^\beta{}_\mu} = i\hbar\, g^{\alpha\beta}$$

commutation relations:

$$[\hat{L}^{\alpha\beta}, x^\gamma]_- = -(L^{\alpha\beta})^\gamma{}_\delta\, x^\delta$$

$$[\hat{L}^{\alpha\beta}, \hat{p}^\gamma]_- = -(L^{\alpha\beta})^\gamma{}_\delta\, \hat{p}^\delta$$

$x^\gamma, \hat{p}^\gamma$: vector operators = tensor operators of rank 1

$\hat{O}^{\lambda_1 \dots \lambda_n}$: tensor of rank n

$$[\hat{L}^{\mu\nu}, \hat{O}^{\lambda_1 \dots \lambda_n}]_- = -\sum_{k=1}^{n} (L^{\mu\nu})^{\lambda_k}{}_{x}\, \hat{O}^{\lambda_1 \dots \lambda_{k-1}\, x\, \lambda_{k+1} \dots \lambda_n}$$

$$[\hat{L}^{\alpha\beta}, \hat{L}^{\gamma\delta}]_- = -(L^{\alpha\beta})^\gamma{}_\sigma\, \hat{L}^{\sigma\delta} - (L^{\alpha\beta})^\delta{}_\sigma\, \hat{L}^{\gamma\sigma}$$

angular momentum operator is a tensor of rank 2

$$[\hat{L}^{\alpha\beta}, \hat{L}^{\gamma\delta}]_- = i\, \underbrace{\boxed{C^{\alpha\beta\gamma\delta}{}_{\varepsilon\xi}}}_{\text{structure coefficients of Lorentz algebra}}\, \hat{L}^{\varepsilon\xi}$$

2.9 Tensor / Spinor Field Representation:

tensor / spinor field: $\psi^{\sigma}(\underbrace{x^{\mu}}_{\text{space-time four vector}})$

tensorial / spinorial quantities