2.9 Tensor / Spinor Field Representation:

scalar field = tensor 0th rank $\quad \phi'(x'^\mu) = \phi(x^\mu)$

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$\Rightarrow \quad \phi'(\underbrace{x^\mu}_{\text{drop }'}) = \phi((\Lambda^{-1})^\mu{}_\nu x^\nu)$

tensor / spinor field: $\quad \Psi^\sigma(\underbrace{x^\mu}_{\text{space-time vector}})$

components

2.9.1 Four-Vector Potential as Example:

$A^\sigma(x^\mu)$

four-vector $\quad$ four-vector

vector = tensor 1st rank

invariance: $\quad A'^\sigma(x'^\mu) = \Lambda^\sigma{}_\tau A^\tau(\underbrace{x^\mu}_{=(\Lambda^{-1})^\mu{}_\nu x'^\nu})$

drop $'$

$\Rightarrow \quad A'^\sigma(x^\mu) = \underbrace{\Lambda^\sigma{}_\tau}_{} A^\tau(\underbrace{(\Lambda^{-1})^\mu{}_\nu x^\nu}_{})$

$= g^\sigma{}_\tau - \frac{i}{2}(L^{\alpha\beta})^\sigma{}_\tau \omega_{\alpha\beta} \qquad = g^\mu{}_\nu + \frac{i}{2}(L^{\alpha\beta})^\sigma{}_\tau \omega_{\alpha\beta}$

expansion to first order: Lorentz algebra

$$A'^\sigma(x^\mu) = \left\{ g^\sigma{}_\tau - \frac{i}{2}\omega_{\alpha\beta}(\hat{M}^{\alpha\beta})^\sigma{}_\tau \right\} A^\tau(x^\mu)$$

total angular momentum operator $\quad \hat{M}^{\alpha\beta} = \underline{\hat{L}^{\alpha\beta}} + \underline{L^{\alpha\beta}}$

$\frac{1}{\hbar}(x^\alpha \hat{p}^\beta - x^\beta \hat{p}^\alpha)$; $\hat{p}^\alpha = i\hbar\partial^\alpha$ = orbital angular momentum operator   spin angular momentum

observation $[\hat{L}^{\alpha\beta}, L^{\gamma\delta}]_- \underset{\uparrow}{=} 0 \Rightarrow \hat{M}^{\alpha\beta}$ is also a representation of Lorentz algebra

independence

$$[\hat{M}^{\alpha\beta}, \hat{M}^{\gamma\delta}]_- = g^{\alpha\delta}\hat{M}^{\beta\gamma} + g^{\beta\gamma}\hat{M}^{\alpha\delta} - g^{\alpha\gamma}\hat{M}^{\beta\delta} - g^{\beta\delta}\hat{M}^{\alpha\gamma}$$

2.9.2 General Case:

framework of a general tensorial / spinorial representation of Lorentz group

$$\psi'^\sigma(x^\mu) = \{ g^\sigma{}_\tau - \frac{i}{2}\omega_{\alpha\beta} \underbrace{\hat{M}^{\alpha\beta}}_{= \hat{L}^{\alpha\beta} + \underbrace{N^{\alpha\beta}}_{\text{not known yet concretely}}} \} \psi^\tau(x^\mu)$$

$$[N^{\alpha\beta}, N^{\gamma\delta}]_- = g^{\alpha\delta}N^{\beta\gamma} + g^{\beta\gamma}N^{\alpha\delta} - g^{\alpha\gamma}N^{\beta\delta} - g^{\beta\delta}N^{\alpha\gamma}$$

$[\hat{L}^{\alpha\beta}, N^{\gamma\delta}]_- = 0 \Rightarrow \hat{M}^{\alpha\beta}$: representation of Lorentz algebra

2.10 Defining Representation of Poincaré Group:

Poincaré transformation = Lorentz transformation $\Lambda^\mu{}_\nu$ + shift a^μ

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

Lorentz transformation: scalar products of four-vectors are invariant
Poincaré " : leaves distances between " invariant

$$g_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) = g_{\mu\nu}(x'^\mu - y'^\mu)(x'^\nu - y'^\nu)$$

Poincaré transformation = inhomogeneous Lorentz transformation

Set P of all Poincaré transformations is a group. An element of P is denoted by (Λ, a):

- Closedness: $(\Lambda_1, a_1), (\Lambda_2, a_2) \in P$

$$x_2^\mu = \Lambda_2{}^\mu{}_\nu x_1^\nu + a_2^\mu = \Lambda_2{}^\mu{}_\nu (\Lambda_1{}^\nu{}_\varkappa x^\varkappa + a_1^\nu) + a_2^\mu$$

$$= \underbrace{\Lambda_2{}^\mu{}_\nu \Lambda_1{}^\nu{}_\varkappa}_{=\Lambda^\mu{}_\varkappa} x^\varkappa + \underbrace{\Lambda_2{}^\mu{}_\nu a_1^\nu + a_2^\mu}_{=a^\mu}$$

short-hand notation: $(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$

P = semi-direct product of Lorentz group and translational group

[direct product: $(\Lambda_2, a_2)(\Lambda_1, a_1)(\Lambda_2\Lambda_1, a_1 + a_2)$]

- associativity: $(\Lambda_1, a_1), (\Lambda_2, a_2), (\Lambda_3, a_3) \in P$

$$(\Lambda_1, a_1)\big(\underbrace{(\Lambda_2, a_2)(\Lambda_3, a_3)}_{=(\Lambda_2\Lambda_3,\, \Lambda_2 a_3 + a_2)}\big) = (\Lambda_1\Lambda_2\Lambda_3, \Lambda_1\Lambda_2 a_3 + \Lambda_1 a_2 + a_1)$$

$$\big((\Lambda_1, a_1)(\Lambda_2, a_2)\big)(\Lambda_3, a_3) = (\Lambda_1\Lambda_2\Lambda_3, \Lambda_1\Lambda_2 a_3 + \Lambda_1 a_2 a_1)$$

$$= (\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1)$$

- unit element of $\mathcal{P}$: $(\Lambda_e, a_e) = (I, 0)$

$(\Lambda, a) \in \mathcal{P}$: $(I,0)(\Lambda,a) = (\Lambda,a) = (\Lambda,a)(I,0)$

- inverse element of $\mathcal{P}$: $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} a)$

$$(\Lambda,a)^{-1}(\Lambda,a) = (\Lambda^{-1}, -\Lambda^{-1}a)(\Lambda,a) = (\underbrace{\Lambda^{-1}\Lambda}_{=I}, \underbrace{\Lambda^{-1}a - \Lambda^{-1}a}_{=0})$$

Remark: $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{P}_3 \oplus \mathcal{P}_4 = \mathcal{P}$

depending on $\det\Lambda$, $\Lambda^0{}_0$; in the following: restrict ourselves $\mathcal{P}_1$

2.11 Tensor / Spinor representation of Poincaré Algebra:

Translation with a shift vector a^μ:

$$x'^\mu = x^\mu + a^\mu \qquad \Leftrightarrow \qquad x^\mu = x'^\mu - a^\mu$$

Passive interpretation: spinor / tensor representation

$$\psi'^\sigma(x'^\mu) = \psi^\sigma(\underbrace{x^\mu}_{= x'^\mu - a^\mu \text{ drop } '}) \quad \Rightarrow \quad \psi'^\sigma(x^\mu) = \psi^\sigma(x^\mu - a^\mu)$$

infinitesimal translation: $a^\mu = \varepsilon^\mu$

$$\psi'^\sigma(x^\mu) = \psi^\sigma(x^\mu - \varepsilon^\mu) \underset{\text{Taylor, up to 1st order}}{=} \psi^\sigma(x^\mu) - \varepsilon^\mu \partial_\mu \psi^\sigma(x^\mu) + \ldots$$

$$= \psi^\sigma(x^\mu) + \frac{i}{\hbar} \varepsilon_\mu \underbrace{\hat{p}^\mu}_{= i\hbar \partial^\mu \text{ four-momentum operator}} \psi^\sigma(x^\mu)$$

result: basis generator of translations = four-momentum operator

Extend this to full Poincaré algebra

$$\psi'^{\sigma}(x^\mu) = \{ 1 - \underbrace{\frac{i}{2}\omega_{\alpha\beta}\hat{M}^{\alpha\beta}}_{\text{infinitesimal Lorentz transformation}} + \underbrace{\frac{i}{\hbar}\varepsilon_\alpha \hat{p}^\alpha}_{\text{infinitesimal translation}} \} \psi^{\sigma}(x^\mu)$$

$$\hat{M}^{\alpha\beta} = \hat{L}^{\alpha\beta} + N^{\alpha\beta}, \quad \hat{L}^{\alpha\beta} = \frac{1}{\hbar}(x^\alpha \hat{p}^\beta - x^\beta \hat{p}^\alpha)$$

Commutator relations between generators of Poincaré algebra:

$[\hat{p}^\alpha, \hat{p}^\beta]_- = 0 \;\hat{=}$ 1) translations are subgroup of Poincaré group 2) abelian

$$[N^{\alpha\beta}, \hat{p}^\gamma]_- = 0 \Rightarrow [\hat{M}^{\alpha\beta}, \hat{p}^\gamma]_- = [\hat{L}^{\alpha\beta}, \hat{p}^\gamma]_- = i(g^{\beta\gamma}\hat{p}^\alpha - g^{\alpha\gamma}\hat{p}^\gamma) \; (*)$$

$$[\hat{M}^{\alpha\beta}, \hat{M}^{\gamma\delta}]_- = i(g^{\alpha\delta}\hat{M}^{\beta\gamma} + g^{\beta\gamma}\hat{M}^{\alpha\delta} - g^{\alpha\gamma}\hat{M}^{\beta\delta} - g^{\beta\delta}\hat{M}^{\alpha\gamma})$$

1) Lorentz group is subgroup of Poincaré group
2) non-abelian

2.12 Casimir operators of Poincaré Algebra:

Casimir operator commutes with all elements of Lie algebra

1. Casimir operator: $\hat{p}^2 = g_{\alpha\beta}\hat{p}^\alpha \hat{p}^\beta$

$$[\hat{A}\hat{B}, \hat{C}]_- = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} = \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} = \hat{A}[\hat{B},\hat{C}]_- + [\hat{A},\hat{C}]_-\hat{B}$$

$$[\hat p^2, \hat p^\alpha]_- = g_{\beta\gamma}[\hat p^\beta \hat p^\gamma, \hat p^\alpha]_- = g_{\beta\gamma}\left(\hat p^\beta [\hat p^\gamma, \hat p^\alpha]_- + [\hat p^\beta, \hat p^\alpha]_- \hat p^\beta\right) \equiv 0$$

$\hat p^2$ is by construction a Lorentz scalar, thus it should commute with Lorentz algebra $\Rightarrow$ $[\hat p^2, m^{\alpha\beta}]_- \overset{\text{1) FTBC-rule}}{\underset{\text{2) }(*)}{=}} \equiv 0$

2. Casimir operator: more difficult to construct

Pauli-Lubanski operator $\quad \hat W_\alpha = \frac{1}{2} \underbrace{\varepsilon_{\alpha\beta\gamma\delta}}\, \hat p^\beta \hat m^{\gamma\delta}$

four-dimensional anti-symmetric unity tensor

$\varepsilon_{1234} = +1$

$$\varepsilon_{\alpha\beta\gamma\delta} = -\varepsilon_{\beta\alpha\gamma\delta} = -\varepsilon_{\gamma\beta\alpha\delta} = -\varepsilon_{\delta\beta\gamma\alpha} = -\varepsilon_{\alpha\gamma\beta\delta} = -\varepsilon_{\alpha\delta\gamma\beta} = -\varepsilon_{\alpha\beta\delta\gamma}$$

Scalar product of $\hat W_\alpha$ and $\hat p^\alpha$ vanishes:

$$\hat W_\alpha \hat p^\alpha = \frac{1}{2} \underbrace{\varepsilon_{\alpha\beta\gamma\delta}}_{\substack{\text{anti-symmetric}\\ \text{in } \alpha,\beta}} \underbrace{\hat p^\alpha \hat p^\beta}_{\text{symmetric in } \alpha,\beta} \hat m^{\gamma\delta} \equiv 0$$

$\hat W^\alpha$ commutes with $\hat p^\sigma$:

$$[\hat W^\alpha, \hat p^\sigma]_- = g^{\alpha\alpha'}[\hat W_{\alpha'}, \hat p^\sigma]_- = \frac{1}{2} g^{\alpha\alpha'} \varepsilon_{\alpha'\beta\gamma\delta} \underbrace{[\hat p^\beta \hat m^{\gamma\delta}, \hat p^\sigma]_-}$$

$$\overset{\text{ABC rule}}{=} \hat p^\beta \underbrace{[\hat m^{\gamma\delta}, \hat p^\sigma]_-}_{= i(g^{\delta\sigma}\hat p^\gamma - g^{\gamma\sigma}\hat p^\delta)} + \underbrace{[\hat p^\beta, \hat p^\sigma]_-}_{=0} \hat m^{\gamma\delta}$$

$$= \frac{i}{2}\, g^{\alpha\alpha'} \underbrace{\varepsilon_{\alpha'\beta\gamma\delta}}_{\text{anti-symm. } \beta,\gamma \atop \beta,\delta} \left(g^{\delta\sigma} \underbrace{\hat p_\sigma \hat p_\beta \hat p^\gamma}_{\text{symm. in } \beta,\gamma} - g^{\gamma\sigma} \underbrace{\hat p_\sigma \hat p_\beta \hat p^\delta}_{\text{symm. in } \beta,\delta} \right) \equiv 0$$

$$[\hat M^{\alpha\beta}, \hat W^\gamma]_- = g^{\gamma\delta} [\hat M^{\alpha\beta}, \hat W_\gamma]_- =$$ 1) AB null 2) def. of Pauli-Lub.

$$= \frac{1}{2} g^{\gamma\delta} \varepsilon_{\delta\rho\sigma\tau} \Big\{ \underbrace{[\hat M^{\alpha\beta}, \hat p^\rho]_-}_{= i(g^{\beta\rho} \hat p^\alpha - g^{\alpha\rho} \hat p^\beta)} \hat M^{\sigma\tau} + \hat p^\rho \underbrace{[\hat M^{\alpha\beta}, \hat M^{\sigma\tau}]_-}_{= i(g^{\alpha\tau} \hat M^{\beta\sigma} + g^{\beta\sigma} \hat M^{\alpha\tau} - g^{\beta\tau} \hat M^{\alpha\sigma} - g^{\alpha\sigma} \hat M^{\beta\tau})} \Big\}$$

$$= \frac{i}{2} g^{\gamma\delta} \varepsilon_{\delta\rho\sigma\tau} \Big\{ g^{\beta\rho} \hat p^\alpha \hat M^{\sigma\tau} - g^{\alpha\rho} \hat p^\beta \hat M^{\sigma\tau}$$

$$+ g^{\alpha\tau} \hat p^\rho \hat M^{\beta\sigma} \pm g^{\beta\sigma} \hat p^\rho \hat M^{\alpha\tau} - g^{\beta\tau} \hat p^\rho \hat M^{\alpha\sigma} \mp g^{\alpha\sigma} \hat p^\rho \hat M^{\beta\tau} \Big\}$$

$\tau \to \rho$, $\rho \to \sigma$, $\sigma \to \tau$ cycl. perm.

$\sigma \to \rho$, $\rho \to \sigma$ change cyclic.

combine

$$= \frac{i}{2} g^{\gamma\delta} \underbrace{\Big\{ g^{\beta\rho} \cdot \varepsilon_{\delta\rho\sigma\tau} (\hat p^\alpha \hat M^{\sigma\tau} - 2 \hat p^\sigma \hat M^{\alpha\tau}) - g^{\alpha\rho} \varepsilon_{\delta\rho\sigma\tau} (\hat p^\beta \hat M^{\sigma\tau} - 2 \hat p^\sigma \hat M^{\beta\tau}) \Big\}}$$

Must be related to the Pauli-Lubanski vector otherwise it would be no vector