Reminder:

$\hat{O}^{\lambda_1 \dots \lambda_n}$: tensor operator of rank n

$$[\hat{M}^{\mu\nu}, \hat{O}^{\lambda_1 \dots \lambda_n}]_- = -\sum_{k=1}^{n} (L^{\mu\nu})^{\lambda_k}{}_{\varkappa}\, \hat{O}^{\lambda_1 \dots \lambda_{k-1} \varkappa \lambda_{k+1} \dots \lambda_n}$$

$\hat{p}^\alpha$: tensor operator of rank $n = 1$ ($\hat{=}$ vector operator)

$$[\hat{M}^{\mu\nu}, \hat{p}^\alpha]_- = -(L^{\mu\nu})^\alpha{}_\varkappa\, \hat{p}^\varkappa = -i(g^{\mu\alpha} g^\nu{}_\varkappa - g^{\nu\alpha} g^\mu{}_\varkappa)\, \hat{p}^\varkappa$$

$$= -i(g^{\mu\alpha} \hat{p}^\nu - g^{\nu\alpha} \hat{p}^\mu)$$

Now back to Pauli-Lubanski vector operator: $\hat{W}_\alpha = \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta}\, \hat{p}^\beta \hat{M}^{\gamma\delta}$

$$[\hat{M}^{\alpha\beta}, \hat{W}^\gamma]_- = \text{" last time "}$$

$$= \frac{i}{2} g^{\gamma\delta} \Big\{ \underbrace{g^{\beta\rho} \cdot \varepsilon_{\delta\rho\sigma\tau} (\hat{p}^\alpha \hat{M}^{\sigma\tau} - 2\hat{p}^\sigma \hat{M}^{\alpha\tau}) - g^{\alpha\rho} \varepsilon_{\delta\rho\sigma\tau} (\hat{p}^\beta \hat{M}^{\sigma\tau} - 2\hat{p}^\sigma \hat{M}^{\beta\tau})}_{\text{must be related to } \hat{W}^\alpha \text{ in order that } \hat{W}^\alpha \text{ is, indeed, a vector operator in the above sense}} \Big\}$$

(3)

$$\varepsilon_{\alpha\beta\cancel{\delta}\delta}\,\varepsilon^{\alpha'\beta'\cancel{\gamma'}\delta}_{\ \ \ \ \gamma} = \delta_\alpha^{\ \alpha'}\delta_\beta^{\ \beta'}\delta_\gamma^{\ \cancel{\gamma'}\gamma} + \delta_\alpha^{\ \beta'}\delta_\beta^{\ \cancel{\gamma'}\gamma}\cancel{\delta_\gamma^{\ \alpha'}} + \delta_\alpha^{\ \cancel{\gamma'}\gamma}\delta_\beta^{\ \alpha'}\delta_\gamma^{\ \beta'}$$

$$-\delta_\alpha^{\ \beta'}\delta_\beta^{\ \alpha'}\delta_\gamma^{\ \cancel{\gamma'}\gamma} - \delta_\alpha^{\ \alpha'}\delta_\beta^{\ \cancel{\gamma'}\gamma}\delta_\gamma^{\ \beta'} - \delta_\alpha^{\ \cancel{\gamma'}\gamma\,\alpha'}\delta_\beta^{\ \beta'}\cancel{\delta_\gamma^{\ \alpha'}}$$

$$\hat{W}_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta'\gamma'\delta'}\hat{p}^{\beta'}\hat{M}^{\gamma'\delta'}$$

$$\hat{W}_\alpha\varepsilon^{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon^{\beta\gamma\delta\alpha}\varepsilon_{\beta'\gamma'\delta'\alpha}\hat{p}^{\beta'}\hat{M}^{\gamma'\delta'}$$

$$= \frac{1}{2}\Big(\hat{p}^\beta\hat{M}^{\gamma\delta} + \hat{p}^\gamma\underline{\hat{M}^{\delta\beta}} + \hat{p}^\delta\hat{M}^{\beta\gamma}$$

$$-\hat{p}^\gamma\underbrace{\hat{M}^{\beta\delta}}_{=-\hat{M}^{\delta\beta}} - \hat{p}^\beta\underbrace{\hat{M}^{\delta\gamma}}_{=-\hat{M}^{\gamma\delta}} - \hat{p}^\delta\underbrace{\hat{M}^{\gamma\beta}}_{-\hat{M}^{\beta\gamma}}\Big) = \hat{p}^\beta\hat{M}^{\gamma\delta} + \hat{p}^\gamma\hat{M}^{\delta\beta} + \hat{p}^\delta\hat{M}^{\beta\gamma} \quad (**)$$

$$\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha'\beta'\gamma\delta} = \underbrace{(4-1-1)}_{=2}\delta_\alpha^{\ \alpha'}\delta_\beta^{\ \beta'} + \underbrace{(1+1-4)}_{=-2}\delta_\alpha^{\ \beta}\delta_\beta^{\ \alpha'} \quad (*)$$

$$\left(\hat{W}_\alpha\varepsilon^{\alpha\beta\gamma\delta}\right)\varepsilon_{\sigma\tau\gamma\delta}$$

1. evaluation with (*)

$$2\hat{W}_\alpha\delta_\sigma^{\ \alpha}\delta_\tau^{\ \beta} - 2\hat{W}_\alpha\delta_\sigma^{\ \beta}\delta_\tau^{\ \alpha} = 2\hat{W}_\sigma\delta^\beta_{\ \tau} - 2\hat{W}_\tau\delta_\sigma^{\ \beta} \quad (1)$$

2. evaluation with (**)

$$\varepsilon_{\sigma\tau\gamma\delta}\Big(\hat{p}^\beta\hat{M}^{\gamma\delta} + \hat{p}^\gamma\underbrace{\hat{M}^{\delta\beta}}_{=-\hat{M}^{\beta\delta}} + \underbrace{\hat{p}^\delta\hat{M}^{\beta\gamma}}_{=-\hat{p}^\gamma\hat{M}^{\beta\delta}}\Big)$$

anti-sym. of ε-tensor

$$= -2\, \hat{p}^{\gamma} \hat{m}^{\beta\delta} \qquad (2)$$

Conclusion: $(1) = (2)$

Use $(1) = (2)$ in (3): we are done

$$[\hat{M}^{\alpha\beta}, \hat{W}^{\gamma}]_- = -(L^{\alpha\beta})^{\gamma}{}_{\delta}\, \hat{W}^{\delta}$$

Pauli-Lubanski vector operator is, indeed, a vector vector in above sense

$\Rightarrow$ $\hat{W}^2 = g_{\alpha\beta} \hat{W}^{\alpha} \hat{W}^{\beta}$ is a tensor operator of zeroth rank

$\Rightarrow$ This will be our second Casimir operator:

$[\hat{P}^{\alpha}, \hat{W}^2]_- = 0$ due to $[\hat{W}^{\alpha}, \hat{P}^{\beta}]_- = 0$ (see last time)

$[\hat{M}^{\alpha\beta}, \hat{W}^2]_- = g_{\gamma\delta} [\hat{M}^{\alpha\beta}, \hat{W}^{\gamma} \hat{W}^{\delta}]_-$

1) ABC rule
2) $\hat{W}^{\gamma}$ is a vector operator $\quad \equiv 0$

Finally: How to physically interpret $\hat{W}^2$?

describe a particle with four-momentum $p = (p^{\mu})$ via a tensor/spinor field $\Psi^{D}(x^{\mu})$ by the eigenvalue problem

$$\hat{p}^\mu \psi^\sigma(x^\mu) = p^\mu \psi^\sigma(x^\mu)$$

1. Casimir operator: $\hat{p}^2 = g_{\mu\nu}\hat{p}^\mu \hat{p}^\nu \Rightarrow \hat{p}^2 \psi^\sigma(x^\mu) = p^2 \psi^\sigma(x^\mu)$

$$p^2 = g_{\mu\nu} p^\mu p^\nu \overset{\text{relativistic}}{\underset{\text{energy-momentum dispersion}}{=}} (Mc)^2 \quad ; \; M = \text{rest mass}$$

2. Casimir operator: $\hat{M}^{\alpha\beta} = \hat{L}^{\alpha\beta} + \underbrace{N^{\alpha\beta}}_{\text{not specified}}$

$$\hat{L}^{\alpha\beta} = \frac{1}{\hbar}(\hat{x}^\alpha \hat{p}^\beta - \hat{x}^\beta \hat{p}^\alpha)$$

$$\hat{W}_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\hat{p}^\beta \hat{M}^{\gamma\delta}$$ (Note: $\hat{L}^{\alpha\beta}$ drops out of the definition)

$$= \frac{1}{6}\varepsilon_{\alpha\beta\gamma\delta}\underbrace{\left(\hat{p}^\beta \hat{L}^{\gamma\delta} + \hat{p}^\gamma \hat{L}^{\delta\beta} + \hat{p}^\delta \hat{L}^{\beta\gamma}\right)} + \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\hat{p}^\beta N^{\gamma\delta}$$

$$= \frac{1}{\hbar}\left[\hat{p}^\beta(x^\gamma \hat{p}^\delta - x^\delta \hat{p}^\gamma) + \hat{p}^\gamma(x^\delta \hat{p}^\beta - x^\beta \hat{p}^\delta) + \hat{p}^\delta(x^\beta \hat{p}^\gamma - x^\gamma \hat{p}^\beta)\right]$$

$$[\hat{p}^\alpha, x^\beta]_- = \hat{p}^\alpha x^\beta - x^\beta \hat{p}^\alpha = i\hbar g^{\alpha\beta}$$

$$= \frac{1}{\hbar}\left[(x^\gamma \hat{p}^\beta + \cancel{i\hbar g^{\beta\gamma}})\hat{p}^\delta + 5 \text{ other terms}\right]$$

$$= \frac{1}{\hbar}\left[\underbrace{(x^\gamma \hat{p}^\beta - x^\beta \hat{p}^\gamma)}_{= -i\hbar g^{\gamma\beta}}\hat{p}^\delta + 4 \text{ other terms}\right] \equiv 0$$

$$\Rightarrow \hat{W}_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\hat{p}^\beta N^{\gamma\delta}$$

eigenvalue problem: $\hat{W}_\alpha \psi^\sigma(x^\mu) = w_\alpha \psi^\sigma(x^\mu)$

$W_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\, p^\beta N^{\gamma\delta}$: two 3 vector

spatial indices

$S_k = \frac{1}{2}\varepsilon_{klm} N^{lm}$

$\vec{S} = (S_1, S_2, S_3) = \underline{(N^{23}, N^{31}, N^{12})}$

spatio-temporal indices

$K_k = N^{0k}$

$\vec{K} = (K_1, K_2, K_3) = (N^{01}, N^{02}, N^{03})$

$$\boxed{W_0 = \frac{1}{2}\underbrace{\varepsilon_{0ijk}}_{=\varepsilon_{ijk}}\, p^i N^{jk} = p^i\left(\frac{1}{2}\varepsilon_{ijk}N^{jk}\right) = p^i S_i = \vec{p}\cdot\vec{S}}$$

$$W_i = \frac{1}{2}\Big(\underbrace{\varepsilon_{i0jk}}_{=-\varepsilon_{0ijk}=-\varepsilon_{ijk}} p^0 N^{jk} + \underbrace{\varepsilon_{ij0k}}_{=-\varepsilon_{ijk0}=-\varepsilon_{0ijk}=-\varepsilon_{ijk}} p^j \underbrace{N^{0k}}_{=K_k} + \underbrace{\varepsilon_{ijk0}}_{=\varepsilon_{0ijk}=\varepsilon_{ijk}} p^j \underbrace{N^{k0}}_{=-N^{0k}}\Big)$$

$$= \frac{1}{2}\left\{-2S_i - (\vec{p}\times\vec{K})_i - (\vec{p}\times\vec{K})_i\right\} = -\left(p^0\vec{S} + \vec{p}\times\vec{K}\right)_i$$

$$\boxed{\vec{W} = (W^i) = p^0\vec{S} + \vec{p}\times\vec{K}, \quad W_\alpha = (W_0, -\vec{W})}$$

rest frame: $p^0 = Mc$, $\vec{p} = \vec{0}$ $\Rightarrow$ $W^0 = 0$, $\vec{W} = Mc\,\vec{S}$

and: $[S_i, S_j]_- = i\,\varepsilon_{ijk} S_k$

Pauli-Lubanski vector represents in the rest frame of the particle as spin momentum

$\Rightarrow$ W^α is a special relativistic generalization of the particle spin in any inertial system

2.13 Irreducible Representations of Poincaré Group:

Eigenvalues of both Casimir operators of Poincaré algebra allow to classify irreducible representations of Poincaré group.

Note: They are infinite-dimensional as momenta are unbounded

1. Casimir operator: $\hat{P}^2 = g_{\alpha\beta}\hat{P}^\alpha \hat{P}^\beta$, $P^2 = g_{\alpha\beta} P^\alpha P^\beta = (mc)^2$

$\Rightarrow$ Two cases: $m > 0$, $m = 0$

2.13.1 Massive Representation ($m > 0$)

2. Casimir operator: $\hat{W}^2 = g_{\alpha\beta}\hat{W}^\alpha \hat{W}^\beta$, $W^2 = g_{\alpha\beta} W^\alpha W^\beta$ is a Lorentz scalar, i.e. it has in all inertial systems the same value

rest frame: $W^0 = 0$, $\vec{W} = mc\vec{S}$, $W^2 = (W^0)^2 - \vec{W}^2 = = -mc\vec{S}^2$

$[S_i, S_j]_- = i\varepsilon_{ijk} S_k \Rightarrow |\vec{S}|^2 = s(s+1)$, $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

$\longrightarrow$ QM I / II

massive representations are characterized by $M > 0$ and spin s which are the fundamental properties of all elementary particles

2.13.2 Massless Representations ($M=0$):

$M=0$: They do not have a rest frame, there exists no Lorentz transformations into a rest frame

If it would be possible: $p^0 = Mc = 0 = \frac{E}{c} \Rightarrow$ no energy, $\vec{p} = \vec{0}$

$\Rightarrow$ need a different procedure

Four vectors p^α, W^α have the following properties

p^α, W^α are light-like four-vectors

$$\left.\begin{array}{l} p^2 = M^2c^2 = 0 \Rightarrow p_\alpha p^\alpha = 0 \Rightarrow (p^0)^2 = \vec{p}^{\,2} \\ W^2 = -M^2c^2 s(s+1) \equiv 0 \rightarrow W_\alpha W^\alpha = 0 \Rightarrow (W^0)^2 = \vec{W}^2 \end{array}\right\} \begin{array}{l} p^\alpha \neq 0, \quad W^\alpha \neq 0 \\ p^0 \neq 0, \quad W^0 \neq 0 \end{array}$$

W^α, p^α are linear dependent: $A p^\alpha + B W^\alpha = 0$

$\alpha = 0$: $B = -\frac{p^0}{W^0} A$; $p^0 \neq 0$ and $W^0 \neq 0$

Also common-operators are linear dependent: $\hat{w}^\alpha = \underbrace{\hat{h}}_{\sim}\ \hat{p}^\alpha$

proportionality operator

$$\underbrace{[\hat{w}^\alpha, \hat{p}^\beta]}_{=0} = [\hat{h}\hat{p}^\alpha, \hat{p}^\beta]_- \overset{ABC\ rule}{=} \hat{h}\underbrace{[\hat{p}^\alpha, \hat{p}^\beta]_-}_{=0} + [\hat{h}, \hat{p}^\beta]_- \hat{p}^\alpha$$

see before

$$\Rightarrow [\hat{h}, \hat{p}^\beta]_- = 0$$

$$\cancel{\underbrace{[\hat{m}^{\alpha\beta}, \hat{w}^\gamma]_-}_{= i(g^{\beta\gamma}\hat{w}^\alpha - g^{\alpha\gamma}\hat{w}^\beta)}} = [\hat{m}^{\alpha\beta}, \hat{h}\hat{p}^\gamma]_- = [\hat{m}^{\alpha\beta}, \hat{h}]\hat{p}^\gamma + \cancel{\hat{h}\underbrace{[\hat{m}^{\alpha\beta}, \hat{p}^\gamma]_-}_{= i(g^{\beta\gamma}\hat{p}^\alpha - g^{\alpha\gamma}\hat{p}^\beta)}}$$

$\hat{w}^\alpha = h\hat{p}^\alpha$ $\quad$ $\hat{w}^\beta = h\hat{p}^\beta$

$$\Rightarrow [\hat{m}^{\alpha\beta}, \hat{h}]_- = 0$$

$\hat{h}$ is now a Casimir operator

eigenvalue: $w^\alpha = h\,p^\alpha \overset{\alpha=0}{\Longrightarrow} h = \dfrac{w^0}{p^0} = \dfrac{\vec{p}\cdot\vec{S}}{|\vec{p}|}$ $\quad$ $w^0 \neq 0$, $p^0 \neq 0$

helicity

projection of particle spin into propagation direction

2.13. Other Representations:

- $p_\mu p^\mu = 0$ and continuous spin
- $p_\mu p^\mu < 0$ tachyons