# 3 Noether Theorem:

1918, Emmy Noether: classical field theory

continuous symmetries $\overset{\text{relation}}{\longleftrightarrow}$ conservation laws

| | |
|---|---|
| time translation invariance | energy conservation |
| spatial translation " | momentum " |
| rotational " | angular " " |
| ⋮ | ⋮ |

aim: general relation how to determine the conservation laws based on continuous for any classical field theory

## 3.1 Invariance:

action: $\mathcal{A} = \mathcal{A}[\psi^\sigma(\cdot)]$ ← functional

function

$$\mathcal{A} = \underbrace{\frac{1}{c}\int_\Omega d^4x}_{=\int dt \int d^3x} \underbrace{\mathcal{L}\left(\psi^\sigma(\cdot);\ \partial_\mu \psi^\sigma(\cdot)\right)}_{\text{Lagrange function } L} \quad ; \quad d^4x = c\,dt\, d^3x$$

Lagrange density

"local field theory"
≙ neglecting higher spatio-temporal derivatives

Transformation: $\quad x^{\prime\lambda} = x^{\prime\lambda}(x^{\tau})$

$$\psi^{\prime\sigma}(x^{\prime\lambda}) = \psi^{\prime\sigma}(\psi^{\tau}(x^{\lambda}))$$

$$\underbrace{\mathcal{A}' = \mathcal{A}'[\psi^{\prime\sigma}(\cdot)]}$$

$$= \frac{1}{c}\int_{\Omega'} d^4x' \, \mathcal{L}\left(\psi^{\prime\sigma}(\cdot), \partial'_\mu \psi^{\prime\sigma}(x^{\prime\lambda})\right)$$

Invariance: $\quad \mathcal{A}[\psi^{\sigma}(\cdot)] = \mathcal{A}'[\psi^{\prime\sigma}(\cdot)]$

## 3.2 Infinitesimal Transformation:

$$x^{\prime\lambda} = x^{\lambda} + \delta x^{\lambda}$$

$$\psi^{\prime\sigma}(x^{\prime\lambda}) = \psi^{\sigma}(x^{\lambda}) + \underset{\sim}{\delta}\psi^{\sigma}(x^{\lambda})$$

total variation: changes in spacetime and in tensorial / spinorial field space

## 3.3 Local Variation:

$$\tilde{\delta}\psi^{\sigma}(x^{\lambda}) = \psi^{\prime\sigma}(x^{\lambda}) - \psi^{\sigma}(x^{\lambda})$$

identical

$$\tilde{\delta}\psi^{\sigma}(x^{\lambda}) = \left\{\psi^{\prime\sigma}(x^{\lambda}) - \psi^{\prime\sigma}(x^{\prime\lambda})\right\} + \left\{\underbrace{\psi^{\prime\sigma}(x^{\prime\lambda}) - \psi^{\sigma}(x^{\lambda})}_{= \delta\psi^{\sigma}(x^{\lambda})}\right\}$$

$$= -\left\{\psi^{\prime\sigma}(\underbrace{x^{\prime\lambda}}_{x^{\lambda}+\delta x^{\lambda}}) - \psi^{\prime\sigma}(x^{\lambda})\right\}$$

$$\approx \delta\psi^\sigma(x^\lambda) - \partial_\mu \psi'^\sigma(x^\lambda)\cdot\delta x^\mu \approx \delta\psi^\sigma(x^\lambda) - \partial_\mu\psi^\sigma(x^\lambda)\,\delta x^\mu$$

first order

neglect higher order terms in the changes

technical advantage of local variation:

$$\partial_\mu \tilde{\delta}\psi^\sigma(x^\lambda) = \tilde{\delta}\,\partial_\mu\psi^\sigma(x^\lambda) \Longleftarrow$$

$$\partial_\mu \delta\psi^\sigma(x^\lambda) \neq \delta\,\partial_\mu\psi^\sigma(x^\lambda)$$

## 3.4 Total Variation of Action:

Problem: before (after) the symmetry transformation one integrates with respect to $\Omega$ ($\Omega'$)

Solution: Interpret the symmetry transformation passively

$\Rightarrow$ the integration volume is the same but once expressed in $x^\lambda$ and once in $x'^\lambda$, $\quad x'^\lambda = x^\lambda + \delta x^\lambda$

Idea: Transform back from $\Omega'$ to $\Omega$

$\Rightarrow$ Jacobi determinant needed

$\downarrow$

$$\frac{d^4x'}{d^4x} = \frac{\partial(x'^\lambda)}{\partial(x^\mu)} = \left| g^\lambda{}_\mu + \frac{\partial \delta x^\lambda}{\partial x^\mu} \right| = \begin{vmatrix} 1+\frac{\partial \delta x^0}{\partial x^0} & \frac{\partial \delta x^0}{\partial x^1} & \cdots \\ \frac{\partial \delta x^1}{\partial x^0} & 1+\frac{\partial \delta x^1}{\partial x^1} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}$$

$$\approx 1 + \frac{\partial \delta x^\mu}{\partial x^\mu}$$

↑ up to first order in variations

expectation: $\mathcal{L}' = \mathcal{L} + \delta\mathcal{L}$

$$\delta\mathcal{A} = \mathcal{A}'[\psi'^\sigma(\cdot)] - \mathcal{A}[\psi^\sigma(\cdot)] = \frac{1}{c}\int_{\Omega'} d^4x'\, \mathcal{L}' - \frac{1}{c}\int_\Omega d^4x\, \mathcal{L}$$

$$= \frac{1}{c}\int_\Omega d^4x \left\{ \underbrace{\left(1+\frac{\partial \delta x^\mu}{\partial x^\mu}\right)\cdot(\mathcal{L}+\delta\mathcal{L})}_{\approx \cancel{\mathcal{L}} + \mathcal{L}\frac{\partial \delta x^\mu}{\partial x^\mu} + \delta\mathcal{L}} - \cancel{\mathcal{L}} \right\} = \frac{1}{c}\int_\Omega d^4x \left\{ \mathcal{L}\frac{\partial \delta x^\mu}{\partial x^\mu} + \underset{\text{total variation}}{\delta\mathcal{L}} \right\}$$

## 3.5 Total and local variational action:

$$\delta\psi^\sigma(x^\lambda) = \tilde{\delta}\psi^\sigma(x^\lambda) + \partial_\mu \psi^\sigma(x^\lambda)\,\delta x^\mu$$

in analogy: $\delta\mathcal{L} = \tilde{\delta}\mathcal{L} + \partial_\mu\mathcal{L}\,\delta x^\mu$

$$\tilde{\delta}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi^\sigma(x^\lambda)}\tilde{\delta}\psi^\sigma(x^\lambda) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\sigma(x^\lambda))}\underbrace{\tilde{\delta}\partial_\mu\psi^\sigma(x^\lambda)}_{=\partial\ \tilde{\delta}\psi^\sigma(x^\lambda),\ \text{see above}}$$

$$= \left\{ \frac{\partial\mathcal{L}}{\partial\psi^\sigma(x^\lambda)} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\sigma(x^\lambda))} \right\} \tilde{\delta}\psi^\sigma(x^\lambda) + \partial_\mu\left[ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\sigma(x^\lambda))}\tilde{\delta}\psi^\sigma(x^\mu) \right]$$

## 3.6 Continuity Equation:

$$\delta \mathcal{A} = \frac{1}{c}\int_\Omega d^4x \left\{ \boxed{\left[\frac{\partial \mathcal{L}}{\partial \psi^\sigma(x^\lambda)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\sigma(x^\lambda))}\right]} \tilde{\delta}\psi^\sigma(x^\lambda) \right.$$

$$\left. + \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\sigma(x^\lambda))} \underbrace{\tilde{\delta}\psi^\sigma(x^\lambda)}_{=\delta\psi^\sigma(x^\lambda) - \partial_\nu \psi^\sigma(x^\lambda)\delta x^\nu} \right] + \underbrace{\left[\mathcal{L}\frac{\partial \delta x^\mu}{\partial x^\mu} + \partial_\mu \mathcal{L}\, \delta x^\mu\right]}_{=\partial_\mu\left[\mathcal{L}\,\delta x^\mu\right]} \right\}$$

// 0

Euler-Lagrange equation:

$$\frac{\delta \mathcal{A}}{\delta \psi^\sigma(x^\lambda)} = \underset{\substack{\uparrow \\ \text{see exercises}}}{\cdots} = \boxed{\frac{\partial \mathcal{L}}{\partial \psi^\sigma(x^\lambda)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\sigma(x^\lambda))} = 0}$$

$0 =$ ↑ Hamilton principle

$$\delta \mathcal{A} = \frac{1}{c}\int_\Omega d^4x\, \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\sigma(x^\lambda))} \delta\psi^\sigma(x^\lambda) - \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\sigma(x^\lambda))}\partial_\nu \psi^\sigma(x^\lambda) - \mathcal{L}\,\delta^\mu{}_\nu\right] \delta x^\nu \right\}$$

Invariance: $\delta \mathcal{A} \equiv 0$

integration volume $\Omega$ is chosen arbitrarily $\Rightarrow \partial_\mu \underbrace{j^\mu(x^\lambda)}_{\text{"current density"}} = 0$

current density $j^\mu$ is additive in $\delta\psi^\sigma$ and $\delta x^\nu$:

$$j^\mu(x^\lambda) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\sigma(x^\lambda))}\delta\psi^\sigma(x^\lambda) - \left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\sigma(x^\lambda))}\partial_\nu \psi^\sigma(x^\lambda) - \mathcal{L}\,\delta^\mu{}_\nu\right\}\delta x^\nu$$

## 3.7 Conserved Quantities:

$$\partial_\mu j^\mu(x^\lambda) = 0 \quad \Rightarrow \quad 0 = \int d^3x\, \partial_\mu j^\mu(x^\lambda) = \int d^3x \left\{ \frac{1}{c} \frac{\partial j^0(\vec{x},t)}{\partial t} + \underset{\downarrow}{\mathrm{div}}\, \vec{j}(\vec{x},t) \right\}$$

Gauß law $\longrightarrow 0$

$$\frac{\partial}{\partial t} \underbrace{\int d^3x\, j^0(\vec{x},t)}_{\text{global conserved quantity}} = - \int d^3x\, \mathrm{div}\, \vec{j}(\vec{x},t) = - \oint d\vec{o}\, \vec{j}(\vec{x},t) \equiv 0$$

## 3.8 Energy-Momentum Tensor:

infinitesimal translation in space-time:

$$\delta x^\lambda = x'^\lambda - x^\lambda = \varepsilon^\lambda$$

$$\delta \psi^\sigma(x^\lambda) = \psi'^\sigma(x'^\lambda) - \psi^\sigma(x^\lambda) \equiv 0$$

$$\Rightarrow \quad \partial_\mu \Theta^{\mu\nu}(x^\lambda) = 0 \quad , \quad \Theta^{\mu\nu}(x^\lambda) = \Theta^\mu{}_\kappa(x^\lambda) g^{\nu\kappa}$$

$$\Theta^\mu{}_\nu(x^\lambda) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\sigma(x^\lambda))} \partial_\nu \psi^\sigma(x^\lambda) - \delta^\mu{}_\nu \mathcal{L} \quad \Leftarrow$$

Legendre transformation from $\mathcal{L}$ to $\mathcal{H}$

conserved energy-momentum:

$\nu = 0$:

$$p^\nu = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = \left( \int d^3x\, \Theta^{0\nu}(\vec{x},t) \right) , \quad \frac{\partial p^\nu}{\partial t} = 0$$

back to the general Noether theorem

$$j^{\mu}(x\lambda) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\sigma(x\lambda))} \delta\psi^\sigma(x\lambda) - (\text{II})^{\mu}{}_{\nu}(x\lambda)\, \delta x^{\nu}$$