$$\mathcal{A} = \frac{1}{c}\int d^4x\, \mathcal{L}\left(\psi^\sigma(x^\lambda), \partial_\mu \psi^\sigma(x^\lambda)\right) \qquad \frac{1}{c}d^4x = dt\, d^3x, \quad dx^0 = c\,dt$$

$$\left.\begin{aligned} x'^\lambda &= x^\lambda + \underline{\delta x^\lambda} \\ \psi'^\sigma(x^\lambda) &= \psi^\sigma(x^\lambda) + \underline{\delta\psi^\sigma(x^\lambda)} \end{aligned}\right\} \text{infinitesimal transformation}$$

invariance $\delta\mathcal{A} = 0 \Rightarrow \partial_\mu j^\mu(x^\lambda) = 0$

current density: $$j^\mu(x^\lambda) = \frac{1}{c}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\sigma(x^\lambda))}\,\underline{\delta\psi^\sigma(x^\lambda)} - \underbrace{\frac{1}{c}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\sigma(x^\lambda))}\partial_\nu\psi^\sigma(x^\lambda) - \delta^\mu{}_\nu\mathcal{L}\right]}_{=\,T^\mu{}_\nu(x^\lambda)}\underline{\delta x^\nu}$$

$\Rightarrow$ should yield conserved quantities

$$\frac{\partial}{\partial t}\underbrace{\int d^3x\, j^0(\vec{x},t)}_{\text{independent of time = conserved}} = 0$$

in the respective proper SI units

$$(p^\nu) = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}, \quad \frac{\partial p^\nu}{\partial t} = 0, \quad p^\nu = \int d^3x\, T^{0\nu}(\vec{x},t)$$

$$E = \int d^3x\, \underbrace{c\,T^{00}(\vec{x},t)}_{=\mathcal{H}}, \quad \mathcal{H} = \cancel{c}\cdot\frac{1}{\cancel{c}}\left[\frac{\partial\mathcal{L}}{\partial\left(\frac{1}{\cancel{c}}\frac{\partial\psi^\sigma}{\partial t}\right)}\frac{1}{\cancel{c}}\frac{\partial\psi^\sigma}{\partial t} - \underbrace{\delta^{00}}_{=1}\mathcal{L}\right]$$

$$\boxed{= \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi^\sigma}{\partial t}\right)}\frac{\partial\psi^\sigma}{\partial t} - \mathcal{L}}$$

σ: Einstein summation convention

$\Rightarrow$ Legendre transformation from $\mathcal{L}$ to $\mathcal{H}$

$$P^i = \int d^3x\, \mathcal{P}^i, \quad \mathcal{P}^i = \textcircled{1}^{0i} = \cancel{\frac{1}{c}} \frac{\partial \mathcal{L}}{\partial \cancel{\frac{1}{c}} \frac{\partial \psi^\sigma}{\partial t}} \underbrace{g^{ij}}_{= -\delta_{ij}} \frac{\partial \psi^\sigma}{\partial x^j} - \underbrace{g^{0i}}_{=0} \mathcal{L}$$

$$\boxed{\mathcal{P}^i = - \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi^\sigma}{\partial t}} \frac{\partial \psi^\sigma}{\partial x^i}}$$

momentum density

Schrödinger field theory:

$$\mathcal{L} = i\hbar \psi^* \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \vec{\nabla} \psi^* \vec{\nabla} \psi$$

$$(\psi^\sigma) = (\psi, \psi^*), \quad \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi^*}{\partial t}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi}{\partial t}} = i\hbar \psi^*$$

$$\mathcal{H} = \underbrace{\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi^*}{\partial t}}}_{=0} \frac{\partial \psi^*}{\partial t} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi}{\partial t}} \frac{\partial \psi}{\partial t} - \mathcal{L} = \cancel{i\hbar \psi^* \frac{\partial \psi}{\partial t}} - \left(\cancel{i\hbar \psi^* \frac{\partial \psi}{\partial t}} - \frac{\hbar^2}{2m} \vec{\nabla} \psi^* \vec{\nabla} \psi \right)$$

$$= + \frac{\hbar^2}{2m} \vec{\nabla} \psi^* \vec{\nabla} \psi$$

$$\mathcal{P}^i = - \underbrace{\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi^*}{\partial t}}}_{=0} \frac{\partial \psi^*}{\partial x^i} - \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi}{\partial t}} \frac{\partial \psi}{\partial x^i} = - i\hbar \psi^* \frac{\partial \psi}{\partial x^i} = \psi^* \underbrace{\left(\frac{\hbar}{i} \frac{\partial}{\partial x^i} \right)}_{= \hat{p}^i} \psi$$

3.9 Angular Momentum:

rotation angles or rapidities

$$\delta x^\lambda = x'^\lambda - x^\lambda \quad = - \frac{i}{2} \omega_{\nu\kappa} (L^{\nu\kappa})^\lambda{}_\mu x^\mu$$

$$\delta \psi^\sigma (x^\lambda) = \psi'^\sigma (x^\lambda) - \psi^\sigma (x^\lambda) = - \frac{i}{2} \omega_{\nu\kappa} (N^{\nu\kappa})^\sigma{}_\tau \psi^\tau (x^\lambda)$$

insert this into Noether theorem: $\partial_\mu \mathcal{J}^{\mu\nu\kappa} (x^\lambda) = 0$

conservation law of angular momentum

$$\boxed{J^{\mu\nu\lambda}(x\lambda) = L^{\mu\nu\lambda}(x\lambda) + S^{\mu\nu\lambda}(x\lambda)}$$

$$L^{\mu\nu\lambda}(x\lambda) = i\, \Theta^{\mu}{}_{\varsigma}(x\lambda)\, (L^{\nu\lambda})^{\varsigma}{}_{\varsigma'}\, x^{\varsigma'} = \Theta^{\mu\lambda}(x\lambda)\, x^{\nu} - \Theta^{\mu\nu}(x\lambda)\, x^{\lambda} = -L^{\mu\lambda\nu}(x\lambda)$$

$$S^{\mu\nu\lambda}(x\lambda) = -\frac{i}{c} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^{\sigma}(x\lambda))} (N^{\nu\lambda})^{\sigma}{}_{\tau}\, \psi^{\tau}(x\lambda) = -S^{\mu\lambda\nu}(x\lambda) \qquad (4)$$

$$\partial_\mu L^{\mu\nu\lambda} + \partial_\mu S^{\mu\nu\lambda} = \underbrace{(\partial_\mu \Theta^{\mu\lambda})}_{=0} x^{\nu} - \underbrace{(\partial_\mu \Theta^{\mu\nu})}_{=0} x^{\lambda} + \underbrace{\Theta^{\mu\lambda} \delta^{\nu}{}_{\mu}}_{=\Theta^{\nu\lambda}} - \underbrace{\Theta^{\mu\nu} \delta^{\lambda}{}_{\mu}}_{=\Theta^{\lambda\nu}} + \partial_\mu S^{\mu\nu\lambda} = 0$$

$$\boxed{\Rightarrow\ \partial_\mu S^{\mu\nu\lambda} = \Theta^{\lambda\nu} - \Theta^{\nu\lambda}}$$

Conserved quantities:

$$M^{\nu\lambda} = \int d^3x\, J^{0\nu\lambda}, \qquad \frac{\partial}{\partial t} M^{\nu\lambda} = 0$$

$$= L^{\nu\lambda} + S^{\nu\lambda}$$

$$L^{jk} = \int d^3x \left\{ \Theta^{0k}(\vec{x},t)\, x^{j} - \Theta^{0j}(\vec{x},t)\, x^{k} \right\}$$

$$S^{jk} = \int d^3x\, \frac{-i}{c} \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi^{\sigma}(\vec{x},t))} (N^{jk})^{\sigma}{}_{\tau}\, \psi^{\tau}(\vec{x},t)$$

3.10 Symmetrization of Energy-Momentum Tensor:

Belifante construction

canonical energy momentum tensor $\Theta^{\mu\nu}$ $\longleftrightarrow$ symmetrized energy momentum tensor $T^{\mu\nu}$

$T^{\nu\lambda} = \Theta^{\nu\lambda} + t^{\nu\lambda}$, $T^{\nu\lambda} = T^{\lambda\nu}$

$\partial_\mu S^{\mu\nu\lambda} = t^{\nu\lambda} - t^{\lambda\nu}$

how to determine $t^{\nu\lambda}$?

ansatz: $t^{\nu\lambda} = \partial_\mu \chi^{\mu\nu\lambda}$, $\chi^{\mu\nu\lambda} = -\chi^{\nu\mu\lambda}$

check that physics does not change

1) $\partial_\nu T^{\nu\lambda} = \underbrace{\partial_\nu \Theta^{\nu\lambda}}_{=0} + \underbrace{\partial_\nu \partial_\mu \chi^{\mu\nu\lambda}}_{=0}$ $= 0$

symmetric anti-sym

2) $\int d^3x\, T^{0\lambda} = \int d^3x\, \Theta^{0\lambda} + \int d^3x\, \underbrace{\partial_\mu \chi^{\mu 0\lambda}}_{= \underbrace{\partial_0 \chi^{00\lambda}}_{=0} + \partial_j \chi^{j0\lambda}} = \int d^3x\, \Theta^{0\lambda} + \underbrace{\int d^3x\, \partial_j \chi^{j0\lambda}}_{=0 \text{ (Gauß)}}$

$\partial_\mu S^{\mu\nu\lambda} = \partial_\mu \chi^{\mu\nu\lambda} - \partial_\mu \chi^{\mu\lambda\nu} \Rightarrow \boxed{S^{\mu\nu\lambda} = \chi^{\mu\nu\lambda} - \chi^{\mu\lambda\nu}}$ (1)

solve this with ansatz: $\chi^{\mu\nu\lambda} = \chi_s^{\mu\nu\lambda} + \chi_a^{\mu\nu\lambda}$ (2)

$\chi_s^{\mu\nu\lambda} = \chi_s^{\mu\lambda\nu}$, $\chi_a^{\mu\nu\lambda} = -\chi_a^{\mu\lambda\nu}$ (3)

(2) in (1) $\Rightarrow$ $\chi_a^{\mu\nu\lambda} = \frac{1}{2} S^{\mu\nu\lambda}$; (3) is guaranteed by (4)

$\chi^{\mu\nu\lambda} + \chi^{\nu\mu\lambda} = \chi_s^{\mu\nu\lambda} + \chi_s^{\nu\mu\lambda} + \chi_a^{\mu\nu\lambda} + \chi_a^{\nu\mu\lambda} \equiv 0$

$$\frac{1}{2} S^{\mu\nu\lambda} + \frac{1}{2} S^{\nu\mu\lambda}$$

$$\Rightarrow \quad \chi_S^{\mu\nu\lambda} + \chi_S^{\nu\mu\lambda} = -\frac{1}{2}\left(S^{\mu\nu\lambda} + S^{\nu\mu\lambda}\right)$$

solved by $\chi_S^{\mu\nu\lambda} = -\frac{1}{2}\left(S^{\nu\mu\lambda} + S^{\lambda\mu\nu}\right) \Leftarrow$ sym.

$$\checkmark \; -\frac{1}{2}\left(S^{\nu\mu\lambda} + \cancel{S^{\lambda\mu\nu}}\right) - \frac{1}{2}\left(S^{\mu\nu\lambda} + \cancel{S^{\lambda\nu\mu}}\right)$$

$$(4)$$

$$\chi^{\mu\nu\lambda} = \chi_a^{\mu\nu\lambda} + \chi_S^{\mu\nu\lambda} = \frac{1}{2}\left(S^{\mu\nu\lambda} + S^{\nu\lambda\mu} - S^{\lambda\mu\nu}\right) \Leftarrow$$

For any field theory with non-zero spin the canonical energy-momentum tensor is not symmetric and Belinfante construction is needed.

3.11 Modified Angular Momentum Tensor:

canonical angular momentum tensor $J^{\mu\nu\lambda} \longleftrightarrow$ modified angular momentum tensor $I^{\mu\nu\lambda}$

$$I^{\mu\nu\lambda} = T^{\mu\lambda} x^\nu - T^{\mu\nu} x^\lambda$$

insert formulas above:

$$I^{\mu\nu\lambda} = J^{\mu\nu\lambda} + \partial_\rho \xi^{\rho\mu\nu\lambda}, \quad \xi^{\rho\mu\nu\lambda} = \chi^{\rho\mu\lambda} x^\nu - \chi^{\rho\mu\nu} x^\lambda = -\xi^{\mu\rho\nu\lambda}$$

Physics does not change:

1) $\partial_\mu I^{\mu\nu\lambda} = \underbrace{\partial_\mu J^{\mu\nu\lambda}}_{=0} + \underbrace{\partial_\mu \partial_\rho}_{\text{sym.}} \underbrace{\xi^{\rho\mu\nu\lambda}}_{\text{ant.}} = 0$

2) $\int d^3x\, I^{0\nu\lambda} = \int d^3x\, J^{0\nu\lambda} + \underbrace{\int d^3x\, \partial_i \xi^{i 0\nu\lambda}}_{\text{Gauß} = 0}$

4 Klein-Gordon Field:

Motivation:

- first relativistic quantum field to be discussed in lecture
- represents a free scalar field which describes in its quantized form particles with spin 0
- Higgs particle H: electrically neutral, interaction of H with other particles gives them a mass
- Pions $\pi^\pm$, π^0, introduced by Yukawa exchange particles for nuclear force, mesons (quark + antiquark) not elementary particles

complex field

real field

Minimally coupling charged pions to Maxwell field = scalar QED

- simpler than QED as not spinor fields are needed
- relativistic generalization of the Ginzburg-Landau theory of superconductivity

Menu:

1) Classical field theory

2) Second quantization

3) Propagator