4 Klein-Gordon Theory:

Spinless, massive particles with charge $\hat{=}$ $\Psi(x^\lambda)$, $\Psi^*(x^\lambda)$

$$\mathcal{A} = \mathcal{A}[\Psi^*(\cdot), \Psi(\cdot)]$$
$$= \frac{1}{c}\int d^4x \; \mathcal{L}\left(\Psi^*(x^\lambda), \partial_\mu \Psi^*(x^\lambda) ; \Psi(x^\lambda), \partial_\mu \Psi(x^\lambda)\right)$$

Condition: $\mathcal{L}$ must be Lorentz invariant and real and bilinear

"manifestly" $\hat{=}$ obvious

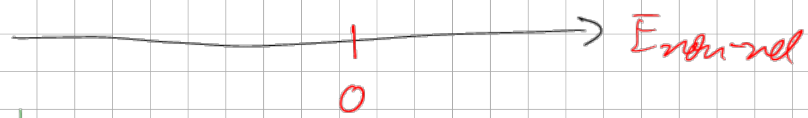
$$\mathcal{L} = A \, g^{\mu\nu} \, \partial_\mu \Psi^*(x^\lambda) \, \partial_\nu \Psi(x^\lambda) + B \, \Psi^*(x^\lambda) \, \Psi(x^\lambda)$$

are not known yet

→ being determined by non-relativistic limit

$$\underbrace{E_{\vec{p}}}_{E_{rel}} = \sqrt{\vec{p}^{\,2} c^2 + m^2 c^4} = mc^2 + \underbrace{\frac{\vec{p}^{\,2}}{2m}}_{E_{non-rel}} + \ldots$$

$$= mc^2 + E_{non-rel}$$



Separation ansatz:

$$\Psi(\vec{x},t) = e^{-\frac{i}{\hbar}mc^2 t}\, \psi(\vec{x},t), \quad \frac{\partial \Psi}{\partial t} = \left\{-\frac{i}{\hbar} mc^2 \psi + \frac{\partial \psi}{\partial t}\right\} e^{-\frac{i}{\hbar}mc^2 t}$$

$$\Psi^*(\vec{x},t) = e^{+\frac{i}{\hbar}mc^2 t}\, \psi^*(\vec{x},t), \quad \frac{\partial \Psi^*}{\partial t} = \left\{+\frac{i}{\hbar} mc^2 \psi^* + \frac{\partial \psi^*}{\partial t}\right\} e^{+\frac{i}{\hbar}mc^2 t}$$

$$\mathcal{L} = A\left\{ \frac{1}{c^2}\frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial t} - \vec\nabla \Psi^* \cdot \vec\nabla \Psi \right\} + B\,\Psi^*\Psi$$

$$= \frac{A}{c^2}\left\{ \underbrace{\frac{\partial \psi^*}{\partial t}\frac{\partial \psi}{\partial t}}_{\substack{A \text{ independent of } c \\ c\to\infty \\ 0}} + \underbrace{\frac{i}{\hbar} m c^2 \left[\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right]}_{\frac{\hbar^2}{2m}\frac{1}{c^2}\frac{imc^2}{\hbar}} \right\} - \underbrace{A\,\vec\nabla\psi^*\cdot\vec\nabla\psi}_{-\frac{\hbar^2}{2m}\vec\nabla\psi^*\cdot\vec\nabla\psi} + \underbrace{\left(B + \frac{m^2c^2}{\hbar^2}A\right)}_{0}\psi^*\psi$$

$$A = \frac{\hbar^2}{2m}$$

$$B = -\frac{m^2c^2}{\hbar^2}A = -\frac{m^2c^2}{\hbar^2}\cdot\frac{\hbar^2}{2m} = -\frac{1}{2}mc^2$$

$$\mathcal{L} = \frac{i\hbar}{\cancel{2}}\cancel{2}\left(\psi^*\frac{\partial \psi}{\partial t} - \cancel{\psi\frac{\partial \psi^*}{\partial t}}\right) - \frac{\hbar^2}{2m}\vec\nabla\psi^*\cdot\vec\nabla\psi \longrightarrow \text{Schrödinger theory}$$

partial integration in time

Result: $$\mathcal{L} = \frac{\hbar^2}{2mc^2}\frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial t} - \frac{\hbar^2}{2m}\vec\nabla\Psi^*\cdot\vec\nabla\Psi - \frac{1}{2}mc^2\,\Psi^*\Psi$$

Hamilton principle:

$$\frac{\delta S}{\delta \Psi^*(\vec x,t)} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \Psi^*(\vec x,t)} - \vec\nabla\frac{\partial \mathcal{L}}{\partial(\vec\nabla\Psi^*(\vec x,t))} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Psi^*(\vec x,t)}{\partial t}\right)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \Psi^*} = -\frac{1}{2}mc^2\Psi, \quad \frac{\partial \mathcal{L}}{\partial(\vec\nabla\Psi^*)} = -\frac{\hbar^2}{2m}\vec\nabla\Psi, \quad \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Psi^*}{\partial t}\right)} = \frac{\hbar^2}{2mc^2}\frac{\partial \Psi}{\partial t}$$

$$-\frac{1}{2}mc^2\Psi + \frac{\hbar^2}{2m}\Delta\Psi - \frac{\hbar^2}{2mc^2}\frac{\partial^2\Psi}{\partial t^2} = 0$$

$$\underbrace{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t}-\Delta\right)}_{=\Box}\Psi+\left(\frac{mc}{\hbar}\right)^2\Psi=0$$

wave equation with additional mass term

$= \Box$ quabla-operator

Compton wave length: $k_c=\frac{mc}{\hbar}=\frac{2\pi}{\lambda_c}\Rightarrow\lambda_c=2\pi\frac{\hbar}{mc}$

$\pi^{\pm}$: $mc^2=139.6\,MeV\Rightarrow\lambda_c=9\,fm$, $1\,fm=10^{-15}\,m$

exchange particle of nuclear force

physical interpretation of Compton wave length:

relativistic particle: $\Delta p=mc$

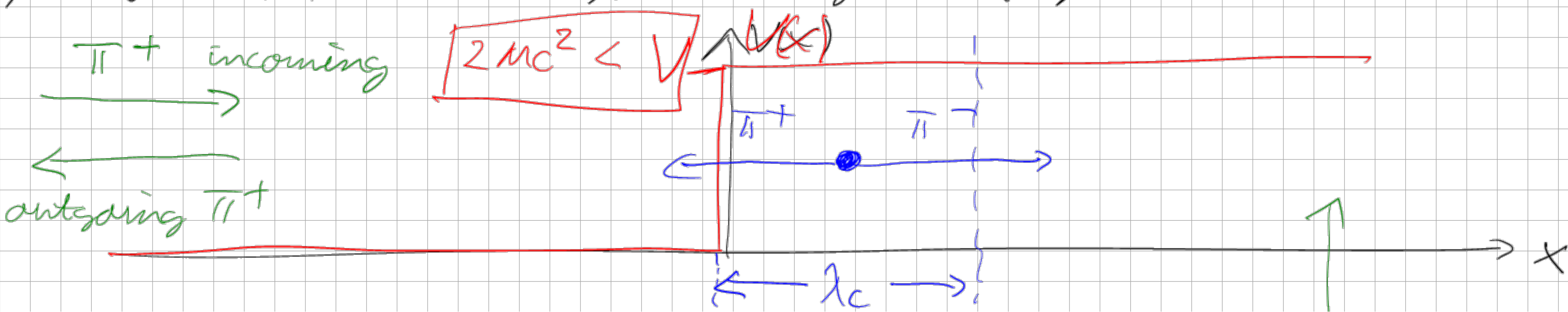
Heisenberg uncertainty: $\Delta x=\frac{\hbar}{\Delta p}$

$$\left.\begin{array}{l}\Delta p=mc\\ \Delta x=\frac{\hbar}{\Delta p}\end{array}\right\}\ \Delta x=\frac{\hbar}{mc}$$

relativistic particle confined in a region of the order of λ_c:

particle-antiparticle creation / annihilation

This phenomenon is best illustrated by Klein paradoxon:



more particles π^+ come out than send in

negatively charged

Moral: Relativistic QFT like the Klein-Gordon theory can never be restricted to a one-particle theory

Non-relativistic limit of Klein-Gordon equation:

$$\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \Delta\Psi + \frac{m^2c^2}{\hbar^2}\Psi = 0 \qquad \Psi = e^{-\frac{i}{\hbar}mc^2 t}\,\psi$$

$$\underbrace{\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}}_{\substack{\downarrow\, c\to\infty \\ =0}} - \frac{2im}{\hbar}\frac{\partial\psi}{\partial t} \underbrace{- \frac{m^2c^4}{\hbar^2}\frac{1}{c^2}\psi + \frac{m^2c^2}{\hbar^2}\psi}_{=0} - \Delta\psi = 0$$

$$\Rightarrow \qquad i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi$$

4.2 Continuity Equation:

$$\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \Delta\Psi + \frac{m^2c^2}{\hbar^2}\Psi = 0 \qquad \Big|\cdot\Psi^*$$

$$-\left(\frac{1}{c^2}\frac{\partial^2\Psi^*}{\partial t^2} - \Delta\Psi^* + \frac{m^2c^2}{\hbar^2}\Psi^* = 0 \qquad \Big|\cdot\Psi\right)$$

$$\frac{1}{c^2}\left(\Psi^*\frac{\partial^2\Psi}{\partial t^2} - \Psi\frac{\partial^2\Psi^*}{\partial t^2}\right) + \underbrace{\Psi\Delta\Psi^* - \Psi^*\Delta\Psi}\, = 0$$

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial t}-\Psi\frac{\partial\Psi^*}{\partial t}\right) = \vec{\nabla}\left(\Psi\vec{\nabla}\Psi^* - \Psi^*\vec{\nabla}\Psi\right)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\rho + \operatorname{div}\vec{j} = 0 \qquad \Big| \cdot k$$

$$\rho = \frac{k}{c^2}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right) \qquad \text{"density"}$$

$$\vec{j} = k\left\{\Psi\vec{\nabla}\Psi^* - \Psi^*\vec{\nabla}\Psi\right\} \qquad \text{"current density"}$$

$$\rho = \frac{k}{c^2}\left\{\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t} - 2i\frac{mc^2}{\hbar}\psi^*\psi\right\} \xrightarrow[c\to\infty]{} \psi^*\psi \Rightarrow k = \frac{i\hbar}{2m}$$

$$\vec{j} = \frac{i\hbar}{2m}\left\{\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right\}$$

Klein-Gordon continuity equation: $\frac{\partial}{\partial t}\rho + \operatorname{div}\vec{j} = 0$

$$\rho = \frac{i\hbar}{2mc^2}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right)$$

$$\vec{j} = \frac{i\hbar}{2m}\left(\Psi\vec{\nabla}\Psi^* - \Psi^*\vec{\nabla}\Psi\right)$$

conserved quantity: $Q = \int d^3x\, \rho(\vec{x},t) \Rightarrow \frac{\partial Q}{\partial t} \equiv 0$

Introduce a "scalar product":

$$\langle\Psi_1,\Psi_2\rangle = \frac{i\hbar}{2mc^2}\int d^3x\left\{\Psi_1^*(\vec{x},t)\frac{\partial\Psi_2(\vec{x},t)}{\partial t} - \Psi_2^*(\vec{x},t)\frac{\partial\Psi_1(\vec{x},t)}{\partial t}\right\}$$

not positive definite:

$$\Psi_1 = \Psi_2 = N e^{\mp \frac{i}{\hbar} mc^2 t} \Rightarrow \langle \Psi, \Psi \rangle = \pm N^2 < 0$$

non-relativistic limit:

$$\langle \Psi_1, \Psi_2 \rangle = \frac{i\hbar}{2mc^2} \int d^3x \left\{ \psi_1^* \frac{\partial \psi_2}{\partial t} - \psi_2^* \frac{\partial \psi_1}{\partial t} - 2i \frac{mc^2}{\hbar} \psi_1^* \psi_2 \right\}$$

$$\xrightarrow{c \to \infty} \underbrace{\frac{i\hbar}{2mc^2} \cdot \frac{-i2mc^2}{\hbar}}_{=1} \int d^3x \, \psi_1^* \psi_2$$

Moral: Each theory has its own natural scalar product

conserved quantity: $Q = \langle \Psi, \Psi \rangle$ can be positive / negative

$eQ \mathrel{\hat{=}}$ charge of π^+, π^-

$Q = 0$ provided $\Psi = \Psi^* \mathrel{\hat{=}} \pi^0, H$

4.3 Application of Noether Theorem:

$$\mathcal{L} = \frac{\hbar^2}{2mc^2} \frac{\partial \Psi^*}{\partial t} \frac{\partial \Psi}{\partial t} - \frac{\hbar^2}{2m} \nabla \Psi^* \nabla \Psi - \frac{mc^2}{2} \Psi^* \Psi$$

Noether: $(\Psi^\sigma) = (\Psi^*, \Psi)$

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi^\sigma}{\partial t}} \frac{\partial \Psi^\sigma}{\partial t} - \mathcal{L} = \underbrace{\frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi^*}{\partial t}}}_{= \pi^*} \frac{\partial \Psi^*}{\partial t} + \underbrace{\frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi}{\partial t}}}_{= \pi} \frac{\partial \Psi}{\partial t} - \mathcal{L}$$

$$\overbrace{= \frac{\hbar^2}{2mc^2}\frac{\partial\Psi}{\partial t}} \qquad \overbrace{\frac{\hbar^2}{2mc^2}\frac{\partial\Psi^*}{\partial t}}$$

$$= \underbrace{\frac{\hbar^2}{2mc^2}\frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial t}}_{=\frac{2mc^2}{\hbar^2}\pi^*\pi} + \frac{\hbar^2}{2m}\vec{\nabla}\Psi^*\cdot\vec{\nabla}\Psi + \frac{mc^2}{2}\Psi^*\Psi \Rightarrow H = \int d^3x\,\mathcal{H}$$

Noether: $$\vec{\mathcal{P}} = -\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Psi^\sigma}{\partial t}\right)}\vec{\nabla}\Psi^\sigma = -\underbrace{\frac{\partial\mathcal{L}}{\partial\frac{\partial\Psi^*}{\partial t}}}_{=\pi^*}\vec{\nabla}\Psi^* - \underbrace{\frac{\partial\mathcal{L}}{\partial\frac{\partial\Psi}{\partial t}}}_{=\pi}\vec{\nabla}\Psi$$

$$= -\frac{\hbar^2}{2mc^2}\left(\frac{\partial\Psi}{\partial t}^*\vec{\nabla}\Psi + \frac{\partial\Psi}{\partial t}\vec{\nabla}\Psi^*\right),$$

$$\underbrace{\vec{P}}_{\text{momentum}} = \int d^3x\,\underbrace{\vec{\mathcal{P}}}_{\text{momentum density}}$$

Internal symmetry: $\delta x^\lambda = x'^\lambda - x^\lambda \equiv 0$

$$\Psi'(x'^\lambda) = \Psi(x^\lambda)\,e^{-i\varphi} \approx (1-i\delta\varphi)\Psi(x^\lambda) \Rightarrow \delta\Psi(x^\lambda) = \Psi'(x'^\lambda) - \Psi(x^\lambda)$$

global phase change

$$= -i\,\Psi(x^\lambda)\,\delta\varphi$$

$$\Psi^{*\prime}(x'^\lambda) = \Psi^*(x^\lambda)\,e^{+i\varphi} \approx (1+i\delta\varphi)\,\Psi(x^\lambda) \Rightarrow \delta\Psi^*(x^\lambda) = +i\,\Psi^*(x^\lambda)\,\delta\varphi$$

Noether theorem: $$\partial_\mu j^\mu = 0 \Rightarrow j^\mu = \frac{1}{c}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^\sigma)}\delta\Psi^\sigma - \Theta^\mu{}_\lambda\,\delta x^\lambda$$

$$j^\mu = \frac{1}{c}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^*)}\underbrace{\delta\Psi^*}_{=+i\Psi^*\delta\varphi} + \frac{1}{c}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\underbrace{\delta\Psi}_{=-i\Psi\delta\varphi} \quad \overset{\sim}{=} 0$$

$$= \frac{i}{c}\left\{\Psi^*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^*)} - \Psi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\right\} \quad ; \quad \partial_\mu j^\mu = 0$$

$$\Rightarrow Q = \frac{1}{\hbar}\int d^3x\, j^0 = \frac{i}{\cancel{\hbar}\cancel{c}}\int d^3x\left\{\Psi^*\underbrace{\frac{\partial\mathcal{L}}{\cancel{\frac{1}{c}}\partial\frac{\partial\Psi^*}{\partial t}}}_{=\pi^*=\frac{\hbar^{\cancel{2}}}{2mc^2}\frac{\partial\Psi}{\partial t}} - \Psi\underbrace{\frac{\partial\mathcal{L}}{\frac{1}{c}\partial\frac{\partial\Psi}{\partial t}}}_{=\pi=\frac{\hbar^{\cancel{2}}}{2mc^2}\frac{\partial\Psi^*}{\partial t}}\right\}$$

additional ↑ constant

$$= \frac{i\hbar}{2mc^2}\int d^3x\left\{\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right\} \longrightarrow \text{charge}$$

final remark: $(j^\mu) = \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}$