Status:

field operator: $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\hat{\vec{A}}(\vec{x},t) = 0$, $\hat{\vec{A}}(\vec{x},t) = \hat{\vec{A}}^+(\vec{x},t)$

$$\hat{\vec{A}}(\vec{x},t) = \int d^3k \left\{ \underbrace{\hat{\vec{A}}(\vec{k})}_{} e^{i(\vec{k}\vec{x} - \omega_{\vec{k}} t)} + \hat{\vec{A}}^+(\vec{k}) e^{-i(\vec{k}\vec{x} - \omega_{\vec{k}} t)} \right\}$$

$$= N_{\vec{k}} \sum_{\underset{\text{helicity}}{\lambda} = \pm 1} \underbrace{\vec{\varepsilon}(\vec{k},\lambda)}_{\text{polarization vector}} \underbrace{\hat{a}_{\vec{k},\lambda}}_{\text{Fourier operators}}$$

$+$ spin $\vec{s} \parallel \vec{k}$ $\qquad$ $-$ spin $\vec{s}$ antiparallel to $\vec{k}$

$$\vec{\varepsilon}(\vec{k},\lambda) = \begin{pmatrix} \cos\vartheta\cos\varphi - \lambda i \sin\varphi \\ \cos\vartheta \sin\varphi + \lambda i \cos\varphi \\ -\sin\vartheta \end{pmatrix}, \quad \vec{k} = k\begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$$

2.10 Properties of Polarization Vectors:

Coulomb gauge: $\operatorname{div}\hat{\vec{A}}(\vec{x},t) = 0 \quad \Leftarrow \quad \vec{k}\cdot\vec{\varepsilon}(\vec{k},\lambda) = 0$

transversality condition

$$\vec{k}\cdot\vec{\varepsilon}(\vec{k},\lambda) = k\{\sin\vartheta\cos\varphi(\cos\vartheta\cos\varphi \cancel{-\lambda i\sin\varphi}) + \sin\vartheta\sin\varphi(\cos\vartheta\sin\varphi \cancel{+\lambda i\cos\varphi}) - \sin\vartheta\cos\vartheta\} \equiv 0 \checkmark$$

1) orthonormality relation:

$$\vec{\varepsilon}^*(\vec{k},\lambda)\cdot\vec{\varepsilon}(\vec{k},\lambda') \overset{\text{exercise}}{=} \delta_{\lambda\lambda'} \quad ; \quad \lambda,\lambda' = \pm 1$$

2) Behaviour under inversion: $\vec{k} \Rightarrow -\vec{k}$

$\varphi \to \varphi + \pi$: $\sin\varphi \to -\sin\varphi$, $\cos\varphi \to -\cos\varphi$

$\vartheta \to \vartheta - \pi$: $\sin\vartheta \to \sin\vartheta$, $\cos\vartheta \to -\cos\vartheta$

$\vec{k} \Rightarrow -\vec{k}$ (spherical coordinates)

$$\left.\right\} \vec{\varepsilon}(-\vec{k},\lambda) \overset{\text{exercise}}{=} \vec{\varepsilon}(\vec{k},-\lambda) = \varepsilon^*(\vec{k},\lambda)$$

$$\hat{\vec{A}}(\vec{x},t) = \sum_{\lambda=\pm 1} \int d^3k\, N_{\vec{k}} \left\{ \vec{\varepsilon}(\vec{k},\lambda) e^{i(\vec{k}\vec{x} - \omega_{\vec{k}} t)} \hat{a}_{\vec{k},\lambda} + \vec{\varepsilon}^*(\vec{k},\lambda) e^{-i(\vec{k}\vec{x} - \omega_{\vec{k}} t)} \hat{a}^+_{\vec{k},\lambda} \right\}$$

What is the physical meaning of these Fourier operators?

2.11 Fourier Operators:

$$\hat{\vec{\pi}}(\vec{x},t) = \varepsilon_0 \frac{\partial \hat{\vec{A}}(\vec{x},t)}{\partial t} = \varepsilon_0 \sum_{\lambda=\pm 1} \int d^3k\, N_{\vec{k}} \left\{ -i\omega_{\vec{k}}\, \vec{\varepsilon}(\vec{k},t)\, e^{i(\vec{k}\vec{x} - \omega_{\vec{k}} t)} \hat{a}_{\vec{k},\lambda} + i\omega_{\vec{k}}\, \vec{\varepsilon}^*(\vec{k},t)\, e^{-i(\vec{k}\vec{x} - \omega_{\vec{k}} t)} \hat{a}^+_{\vec{k},\lambda} \right\}$$

Note: $[\hat{A}_i(\vec{x},t), \hat{\pi}_j(\vec{x}',t)] = i\hbar\, \underbrace{\delta^T_{ij}(\vec{x}-\vec{x}')}_{\text{transversal delta function}}$; $[\hat{A}_i(\vec{x},t), \hat{A}_{i'}(\vec{x}',t)]_- = 0 = [\hat{\pi}_i(\vec{x},t), \hat{\pi}_j(\vec{x}',t)]_-$

exercise

$$\hat{a}_{\vec{k},\lambda} = \frac{1}{2(2\pi)^3 N_{\vec{k}}} \int d^3x\, \vec{\varepsilon}^*(\vec{k},\lambda) \cdot e^{-i(\vec{k}\vec{x} - \omega_{\vec{k}} t)} \left\{ \hat{\vec{A}}(\vec{x},t) + \frac{i}{\varepsilon_0 \omega_{\vec{k}}} \hat{\vec{\pi}}(\vec{x},t) \right\}$$

$$\hat{a}^+_{\vec{k},\lambda} = \frac{1}{2(2\pi)^3 N_{\vec{k}}} \int d^3x\, \vec{\varepsilon}(\vec{k},\lambda) \cdot e^{+i(\vec{k}\vec{x} - \omega_{\vec{k}} t)} \left\{ \hat{\vec{A}}(\vec{x},t) - \frac{i}{\varepsilon_0 \omega_{\vec{k}}} \hat{\vec{\pi}}(\vec{x},t) \right\}$$

$$[\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}]_- = \dots = 0 = [\hat{a}^+_{\vec{k},\lambda}, \hat{a}^+_{\vec{k}',\lambda'}]_-$$

$$\left[\hat{a}_{\vec{k},\lambda},\hat{a}^{+}_{\vec{k}',\lambda'}\right]_- = \underset{\substack{\uparrow\\ \text{transversal delta function}}}{-\,\ldots} = \underbrace{\frac{\hbar}{2(2\pi)^3\varepsilon_0\,\omega_{\vec{k}}\,N_{\vec{k}}^2}}_{\overset{!}{=}1\ \Rightarrow\ N_{\vec{k}}=\sqrt{\frac{\hbar}{2(2\pi)^3\varepsilon_0\,\omega_{\vec{k}}}}}\,\delta(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}$$

represent annihilation and creation operators of bosonic particles, which are characterised by $\vec{k},\lambda$

$\Rightarrow$ photons

2.12 Energy:

$$\hat{\vec{A}}(\vec{x},t)=\sum_{\lambda=\pm1}\int d^3k\sqrt{\frac{\hbar}{2(2\pi)^3\varepsilon_0\,\omega_{\vec{k}}}}\left\{\vec{\varepsilon}(\vec{k},\lambda)e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\hat{a}_{\vec{k},\lambda}+\vec{\varepsilon}^{\,*}(\vec{k},\lambda)e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\hat{a}^{+}_{\vec{k},\lambda}\right\}$$

$$\hat{\vec{\pi}}(\vec{x},t)=\sum_{\lambda=\pm1}\int d^3k\sqrt{\frac{\hbar\,\omega_{\vec{k}}\,\varepsilon_0}{2(2\pi)^3}}\left\{-i\,\vec{\varepsilon}(\vec{k},\lambda)e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\hat{a}_{\vec{k},\lambda}+i\,\vec{\varepsilon}^{\,*}(\vec{k},\lambda)e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\hat{a}^{+}_{\vec{k},\lambda}\right\}$$

$$\hat{H}=\frac{1}{2}\int d^3x\left\{\underbrace{\frac{1}{\varepsilon_0}\hat{\pi}_k(\vec{x},t)\,\hat{\pi}_k(\vec{x},t)}_{\hat{=}\ \text{electric energy}}+\underbrace{\frac{1}{\mu_0}\partial_j\hat{A}_k(\vec{x},t)\,\partial_j\hat{A}_k(\vec{x},t)}_{\hat{=}\ \text{magnetic energy}}\right\}$$

exercise

$$=\ldots=\frac{1}{2}\sum_{\lambda=\pm1}\int d^3k\ \hbar\underbrace{\omega_{\vec{k}}}_{=c|\vec{k}|}\left\{\hat{a}^{+}_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda}+\hat{a}_{\vec{k},\lambda}\hat{a}^{+}_{\vec{k},\lambda}\right\}$$

- Time independent despite of having time dependent $\hat{\vec{A}}(\vec{x},t)$, $\hat{\vec{\pi}}(\vec{x},t)$ due to conservation of energy
- Electromagnetic field consists of independent harmonic oscillators: each energy quantum $\omega_{\vec{k}}$ is doubly degenerate due to $\lambda=\pm1$

vacuum state: $\hat{a}_{\vec{k},\lambda}|0\rangle = 0 = \langle 0|\hat{a}^{+}_{\vec{k},\lambda}$ divergent

vacuum energy: $\langle 0|\hat{H}|0\rangle = \frac{1}{2}\underbrace{\sum_{\lambda=\pm1}}_{=2}\underbrace{\int d^3k\, \hbar\omega_{\vec{k}}}_{\infty}\underbrace{\delta(\vec{0})}_{\infty} = \infty$

renormalized Hamilton operator:

$$:\hat{H}: = \hat{H} - \langle 0|\hat{H}|0\rangle = \cancel{\frac{1}{2}}\sum_{\lambda=\pm1}\hbar\omega_{\vec{k}}\left(\hat{a}^{+}_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda} + \underbrace{\hat{a}_{\vec{k},\lambda}\hat{a}^{+}_{\vec{k},\lambda}}_{=\cancel{\hat{a}^{+}_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda}} + \cancel{\int d^3k\,\hbar\omega_{\vec{k}}\delta(\vec{0})}}\right) - \cancel{\int d^3k\,\hbar\omega_{\vec{k}}\delta(\vec{0})}$$

$$= \sum_{\lambda=\pm1}\int d^3k\,\hbar\omega_{\vec{k}}\underbrace{\hat{a}^{+}_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda}}_{=\hat{n}_{\vec{k},\lambda}}$$ photon number operator

normal ordering

→ creation operators to the left, annihilation operators to the right

2.13 Momentum:

Emmy Noether theorem: $\vec{P} = \int d^3x\, \frac{\vec{S}(\vec{x},t)}{c^2}$

Poynting vector: $\vec{S}(\vec{x},t) = \frac{1}{\mu_0}\underbrace{\vec{E}(\vec{x},t)}_{=-\vec{\pi}} \times \underbrace{\vec{B}(\vec{x},t)}_{\mathrm{rot}\,\vec{A}}$

$$\Rightarrow \hat{\vec{P}} = \int d^3x \left[\vec{\nabla}\times\hat{\vec{A}}(\vec{x},t)\right]\times\hat{\vec{\pi}}(\vec{x},t)$$

→ plane wave decomposition

$$\hat{\vec{P}} = \ldots = \sum_{\lambda=\pm1}\int d^3k\,\frac{\hbar\vec{k}}{2}\left\{\hat{a}^{+}_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda} + \hat{a}_{\vec{k},\lambda}\hat{a}^{+}_{\vec{k},\lambda}\right\}$$

$$\langle 0|\hat{\vec{P}}|0\rangle = \underbrace{\int d^3k\,\hbar\vec{k}}_{=0}\,\delta(\vec{0}) = 0$$

$$:\hat{\vec{P}}: = \hat{\vec{P}} = \sum_{\lambda=\pm1} \int d^3k \, \underline{\hbar \vec{k}} \, \hat{a}^\dagger_{\vec{k}\lambda} \hat{a}_{\vec{k}\lambda}$$

2.14 Spin Angular Momentum:

Noether ~~theorem~~:

$$\hat{\vec{S}} = \int d^3x \, \hat{\vec{A}}(\vec{x},t) \times \hat{\vec{\pi}}(\vec{x},t) \qquad \hat{=} \vec{L} = \vec{x} \times \vec{p}$$

$$= \ldots = \sum_{\lambda=\pm1} \int d^3k \, \lambda \frac{\hbar}{2} \frac{\vec{k}}{|\vec{k}|} \left(\hat{a}^\dagger_{\vec{k}\lambda} \hat{a}_{\vec{k}\lambda} + \hat{a}_{\vec{k}\lambda} \hat{a}^\dagger_{\vec{k}\lambda} \right)$$

$$\langle 0 | \vec{S} | 0 \rangle = \hbar \underbrace{\left(\sum_{\lambda=\pm1} \lambda \right)}_{=0} \underbrace{\left(\int d^3k \frac{\vec{k}}{|\vec{k}|} \right)}_{=0} \delta(\vec{0}) = 0$$

$$:\hat{\vec{S}}: = \hat{\vec{S}} = \sum_{\lambda=\pm1} \int d^3k \, \lambda \hbar \frac{\vec{k}}{|\vec{k}|} \, \hat{a}^\dagger_{\vec{k}\lambda} \hat{a}_{\vec{k},\lambda}$$

helicity

photon has spin 1

2.15 Fock Basis:

first quantized harmonic oscillator (Appendix A)	second quantized electromagnetic field
lowering / raising operators $\hat{a}$, $\hat{a}^+$	annihilation / creation operators $\hat{a}_{\vec{k},\lambda}$, $\hat{a}^\dagger_{\vec{k},\lambda}$
$[\hat{a}, \hat{a}]_- = [\hat{a}^\dagger, \hat{a}^\dagger]_- = 0$	$[\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}]_- = 0 = [\hat{a}^\dagger_{\vec{k},\lambda}, \hat{a}^\dagger_{\vec{k}',\lambda'}]_-$

$$[\hat a, \hat a^+]_- = 1$$

$$[\hat a_{\vec k,\lambda}, \hat a^+_{\vec k',\lambda'}]_- = \delta(\vec k - \vec k')\,\delta_{\lambda\lambda'}$$

---

ground state $|0\rangle$

$$\hat a|0\rangle = 0 = \langle 0|\hat a^+$$

vacuum state $|0\rangle$

$$\hat a_{\vec k,\lambda}|0\rangle = 0 = \langle 0|\hat a^+_{\vec k,\lambda}$$

---

eigenstates of $\hat H$

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat a^+)^n|0\rangle$$

eigenstates of $\hat H$

$$|\{n_{\vec k,\lambda}\}\rangle = \prod_{\vec k,\lambda} \underbrace{|n_{\vec k,\lambda}\rangle}_{\frac{1}{\sqrt{n_{\vec k,\lambda}!}}(\hat a^+_{\vec k,\lambda})^{n_{\vec k,\lambda}}|0\rangle}$$

set of all photon numbers

Fock states

---

orthonormal:

$$\langle n|n'\rangle = \delta_{n,n'}$$

orthonormality

$$\langle\{n_{\vec k,\lambda}\}|\{n'_{\vec k,\lambda}\}\rangle = \prod_{\vec k,\lambda}\delta_{n_{\vec k,\lambda},\, n'_{\vec k,\lambda}}$$

---

completeness:

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$$

completeness

$$\sum_{\lambda=\pm 1}\int d^3k \sum_{n_{\vec k,\lambda}=0}^{\infty} |\{n_{\vec k,\lambda}\}\rangle\langle\{n_{\vec k,\lambda}\}| = 1$$

---

first quantized Hilbert space

$$\mathcal{H} = \mathrm{Span}\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$$

second quantized Fock space

$$\mathcal{F} = \mathrm{Span}\{|0\rangle, |1_{\vec k,\lambda}\rangle, |1_{\vec k,\lambda}; 1_{\vec k',\lambda'}\rangle, |2_{\vec k,\lambda}\rangle, \ldots\}$$

first quantized Hamilton operator

$$\hat{H} = \hbar\omega\Big(\underbrace{\hat{n}}_{=\hat{a}^{+}\hat{a}} + \frac{1}{2}\Big)$$

second quantized Hamilton operator

$$:\hat{H}: = \sum_{\lambda=\pm 1}\int d^3k\, \hbar\,\omega_{\vec{k}}\, \underbrace{\hat{a}^{+}_{\vec{k},\lambda}\,\hat{a}_{\vec{k},\lambda}}_{=\hat{n}_{\vec{k},\lambda}}$$