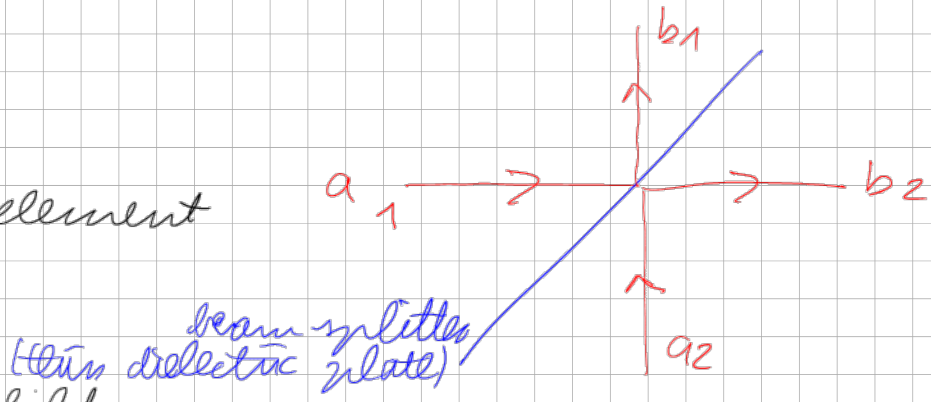
Beam splitter:

- relation to problem 9
- prominent example for an optical element



1. Experimental Set Up:

a_1, a_2: amplitudes of incoming electric field

b_1, b_2: " " outgoing " "

simplest model: linear coupling of modes

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \Rightarrow \quad \vec{b} = \underbrace{U}_{} \ \vec{a}$$

beam splitter matrix (complex!)

2. Loss-Less Beam Splitter:

conservation of energies:

$$|a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2 \quad \Rightarrow \quad \vec{b}^{+} \vec{b} = \vec{a}^{+} \underbrace{U^{+} \ U}_{=1} \vec{a} \quad = \vec{a}^{+} \vec{a}$$

unitary beam splitter matrix $\Leftarrow$ $= 1$

Consequences?

$$|b_1|^2 + |b_2|^2 = (U_{11}^* a_1^* + U_{12}^* a_2^*)(U_{11} a_1 + U_{12} a_2) + (U_{21}^* a_1^* + U_{22}^* a_2^*)(U_{21} a_1 + U_{22} a_2) \overset{!}{=} a_1^* a_1 + a_2^* a_2 + 0 \cdot a_1^* a_2 + 0 \cdot a_1 a_2^*$$

$$\Rightarrow |U_{11}|^2 + |U_{21}|^2 = 1 = |U_{12}|^2 + |U_{22}|^2$$

$$U_{11}^* U_{12} + U_{21}^* U_{22} = 0 = U_{12}^* U_{11} + U_{22}^* U_{21}$$

$U_{ij} = |U_{ij}|\, e^{i\varphi_{ij}}$ solution ansatz

$$\Rightarrow |U_{11}|\, e^{-i\varphi_{11}}\, |U_{12}|\, e^{i\varphi_{12}} + |U_{21}|\, e^{-i\varphi_{21}}\, |U_{22}|\, e^{i\varphi_{22}} = 0$$

$$|U_{11}|\,|U_{12}| = |U_{21}|\,|U_{22}|\, e^{i(\varphi_{11}+\varphi_{22}-\varphi_{12}-\varphi_{21})} \qquad \boxed{\underbrace{(-1)} = e^{\pm i\pi}}$$

$$\Rightarrow \quad |U_{21}| = \frac{|U_{11}|\,|U_{12}|}{|U_{22}|} \quad \text{insert in green relation}$$

$$|U_{11}|^2 \left(\frac{\overbrace{|U_{12}|^2 + |U_{22}|^2}^{=1}}{|U_{22}|^2} \right) = 1 \quad \Rightarrow \quad \frac{|U_{11}|}{|U_{22}|} = 1 = \frac{|U_{21}|}{|U_{12}|}$$

definition: $U_{11} = r\, e^{i\varphi_{11}}$ $\quad U_{22} = r\, e^{i\varphi_{22}}$ $\quad$ with $r^2 + t^2 = 1$

$\qquad U_{12} = t\, e^{i\varphi_{12}}$ $\quad U_{21} = t\, e^{i\varphi_{21}}$

r: reflection coefficient, t: transmission coefficient

3. Symmetric (50:50) Beam Splitter

$$r = t = \frac{1}{\sqrt{2}}, \quad \frac{1}{2} + \frac{1}{2} = 1 \checkmark \qquad \Rightarrow U = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi_{11}} & e^{i\varphi_{12}} \\ e^{i\varphi_{21}} & e^{i\varphi_{22}} \end{pmatrix}$$

4 phases with 1 condition: $\varphi_{11} + \varphi_{22} - \varphi_{12} - \varphi_{21} = \pm\pi$

simplifying assumption:

$\varphi_{12} = \varphi_{21} = 0$: no phase jumps due to transmission

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi_{11}} & 1 \\ 1 & e^{i\varphi_{22}} \end{pmatrix}$$

$$\varphi_{11} + \varphi_{22} = \pm\pi$$

Most common 50:50 beam splitters:

- real unitary matrix: dielectric beam splitter

$\varphi_{11} = 0, \; \varphi_{22} = \mp \pi$ $\qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

- symmetric choice of phases:

$\varphi_{11} = \varphi_{22} = \frac{\pi}{2}$ $\qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$

"symmetric" beam splitter of Loudon

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4. Transition to Quantum Mechanics:

classical amplitudes $\longrightarrow$ second quantized operators

$a_1, a_2; \; b_1, b_2$ $\qquad \hat{a}_1, \hat{a}_2; \; \hat{b}_1, \hat{b}_2$

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

2.16 Quantum Fluctuations of Electric Field:

$$\hat{\vec{A}}(\vec{x},t) = \sum_{\lambda=\pm1} \int d^3k \sqrt{\frac{\hbar}{2(2\pi)^3 \varepsilon_0 \omega_{\vec{k}}}} \left\{ \vec{\varepsilon}(\vec{k},\lambda) e^{i(\vec{k}\vec{x} - \omega_{\vec{k}} t)} \hat{a}_{\vec{k},\lambda} + h.c. \right\}$$

$$\hat{\vec{E}}(\vec{x},t) = -\frac{\partial \hat{\vec{A}}(\vec{x},t)}{\partial t} = \sum_{\lambda=\pm1} \int d^3k \sqrt{\frac{\hbar \omega_{\vec{k}}}{2(2\pi)^3 \varepsilon_0}} \left\{ i\, \vec{\varepsilon}(\vec{k},\lambda) e^{i(\vec{k}\vec{x} - \omega_{\vec{k}} t)} \hat{a}_{\vec{k},\lambda} + h.c. \right\}$$

$$\langle 0 | \hat{\vec{E}}(\vec{x},t) | 0 \rangle = 0 \qquad \hat{a}_{\vec{k},\lambda} |0\rangle = 0 = \langle 0| \hat{a}^+_{\vec{k},\lambda}$$

$$\langle 0 | \underline{\hat{E}_i(\vec{x},t)} \; \underline{\hat{E}_{i'}(\vec{x}',t')} | 0 \rangle = \langle 0 | \left(\hat{a} - \cancel{\hat{a}^+} \right) \left(\cancel{\hat{a}} - \hat{a}^+ \right) | 0 \rangle$$

$$= \sum_{\lambda=\pm1} \sum_{\lambda'=\pm1} \int d^3k \int d^3k' \sqrt{\frac{\hbar\, \omega_{\vec{k}}}{2(2\pi)^3 \varepsilon_0 \cancel{\omega_{\vec{k}}}}} \sqrt{\frac{\hbar\, \omega_{\vec{k}'}}{2(2\pi)^3 \varepsilon_0 \cancel{\omega_{\vec{k}'}}}} \underbrace{i(-i)}_{=1} \varepsilon_i(\vec{k},\lambda)\, \varepsilon^*_{i'}(\vec{k}',\lambda')$$

$$e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\; e^{-i(\vec{k}'\vec{x}'-\omega_{\vec{k}'}t')}\; \langle 0|\underbrace{\hat{a}_{\vec{k},\lambda}\hat{a}^+_{\vec{k}',\lambda'}-\hat{a}^+_{\vec{k}',\lambda'}\hat{a}_{\vec{k},\lambda}}|0\rangle$$

$$=[\hat{a}_{\vec{k},\lambda},\hat{a}^+_{\vec{k}',\lambda'}]-$$

$$=\delta(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}$$

$$=\sum_{\lambda=\pm1}\int d^3k\,\frac{\hbar\,\omega_{\vec{k}}}{2(2\pi)^3\varepsilon_0\cancel{\omega_{\vec{k}}}}\,\varepsilon_i(\vec{k},\lambda)\,\varepsilon_i^*(\vec{k},\lambda)$$

$$\cdot\; e^{i[\vec{k}\underbrace{(\vec{x}-\vec{x}')}-\omega_{\vec{k}}\underbrace{(t-t')}]}$$

homogeneity in space homogeneity in time

Specialization:

isotropy of vacuum

$$\langle 0|\hat{\vec{E}}(\vec{x},t)\hat{\vec{E}}(\vec{x}',t')|0\rangle=\cancel{\sum_{\lambda=\pm1}}2\int d^3k\,\frac{\hbar\,\omega_{\vec{k}}}{\cancel{2}(2\pi)^3\varepsilon_0\cancel{\omega_{\vec{k}}}}\,\underbrace{\vec{\varepsilon}^+(\vec{k},\lambda)\cdot\vec{\varepsilon}(\vec{k},\lambda)}_{=1}$$

$$\cdot\; e^{i[\vec{k}(\vec{x}-\vec{x}')-\omega_{\vec{k}}(t-t')]}$$

spherical coordinates

$$=\int_0^\infty dk\,k^2\int_0^\pi d\vartheta\,\sin\vartheta\underbrace{\int_0^{2\pi}d\varphi}_{=2\pi}\,\frac{\hbar\,\omega_{\vec{k}}}{(2\pi)^3\varepsilon_0\cancel{\omega_{\vec{k}}}}\,e^{i[k\|\vec{x}-\vec{x}'\|\cos\vartheta-\omega_k(t-t')]}$$

$$=\frac{\hbar c}{4\pi^2 i\,\varepsilon_0|\vec{x}-\vec{x}'|}\int_0^\infty dk\,k^2\left\{e^{-ik\underline{[c(t-t')-|\vec{x}-\vec{x}'|]}}-e^{-ik[c(t-t')+|\vec{x}-\vec{x}'|]}\right\}$$

Integral solved by using Gamma function:

$$\Gamma(x)=\int_0^\infty dt\,t^{x-1}e^{-t}\;;\quad x>0$$

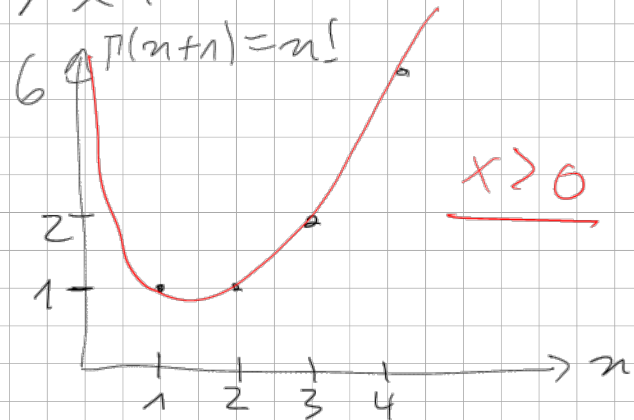
partial integration:

$$\Gamma(x+1) = \int_0^\infty dt\, \underset{u}{t^x}\, \underset{v'}{e^{-t}} = \underbrace{\left[-t^x e^{-t}\right]_0^\infty}_{\underline{x>0}\ 0} - \int_0^\infty dt\,(-e^{-t})\, x\, t^{x-1}$$

$$= x \cdot \Gamma(x) \quad \text{recursion relation}$$

$$\Gamma(n+1) = n \cdot \Gamma(n) = n(n-1)\,\Gamma(n-1) = \ldots = n! \cdot \Gamma(1)$$

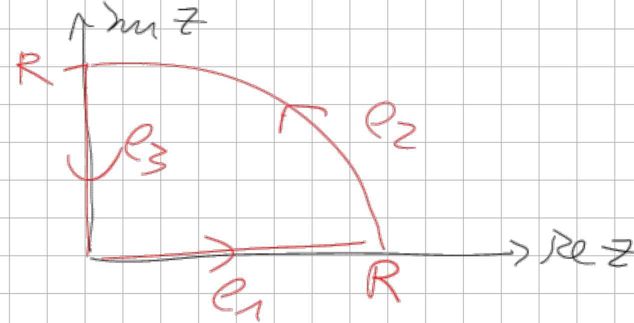
$$\Gamma(1) = \int_0^\infty dt\, e^{-t} = 1$$



x > 0

$$\int_{C_1} dz\, z^{x-1} e^{-z} + \underbrace{\int_{C_2} dz\, z^{x-1} e^{-z}}_{R\to\infty\ =0} + \int_{C_3} dz\, z^{x-1} e^{-z} = 0$$

residue theorem



C_1: $z(t) = t$, $t \in [0, \infty)$

C_3: $z(t) = it$, $t \in (\infty, 0]$

$$\Rightarrow \Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t} = i^x \int_0^\infty dt\, t^{x-1} e^{-it}$$

$t = a\tau$

$$\Rightarrow \int_0^\infty d\tau\, \tau^{x-1} e^{-ia\tau} = \frac{\Gamma(x)}{(ia)^x}$$

Here: $x = 3$, $a = c(t-t') \mp |\vec{x} - \vec{x}'|$

$$\langle 0|\hat{\vec{E}}(\vec{x},t)\cdot\hat{\vec{E}}(\vec{x}',t')|0\rangle = \frac{\hbar c}{2\pi^2 \varepsilon_0 |\vec{x}-\vec{x}'|}\left\{\frac{1}{c(t-t') + |\vec{x}-\vec{x}'|} - \frac{1}{c(t-t') - |\vec{x}-\vec{x}'|}\right\}$$

- such correlations are difficult to measure because intensity measure–

ments are not possible, measurements missing for decades
- Recent experiment: Nature 568, 202 (2019)

$\longrightarrow$ possible topic for student talk

electro-optic detection in nonlinear crystal placed in cryogenic environment

limit $\vec{x}' \to \vec{x}$: de l'Hôpital

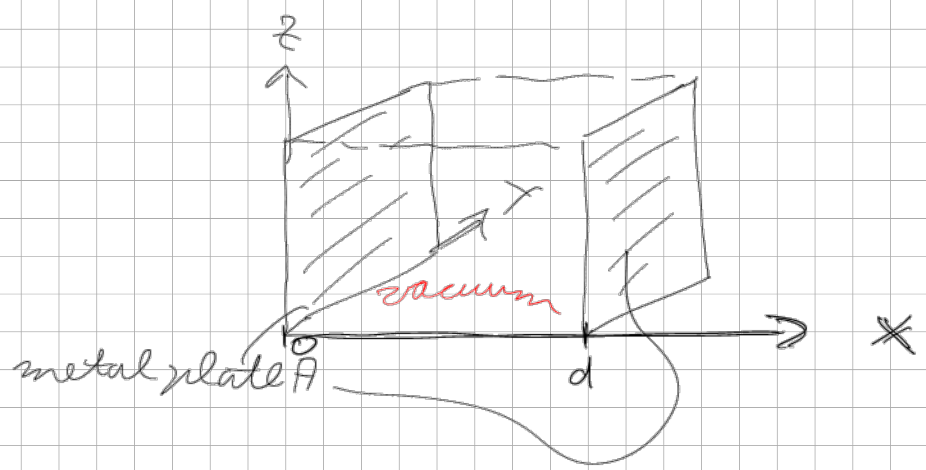
$$\langle 0 | \hat{\vec{E}}(\vec{x},t) \cdot \hat{\vec{E}}(\vec{x},t') | 0 \rangle = \frac{3\hbar}{\pi^2 \varepsilon_0 c^3 (t-t')^4} \xrightarrow{t-t' \to 0} \infty$$

$t \neq t'$: finite correlation

$t = t'$: infinite $\hat{=}$ infinite energy of electromagnetic field in vacuum

2.17 Casimir Effect (1948)

$\Rightarrow$ two plane-parallel metal plates attract each other



2.17.1 Electromagnetic Modes:

- solve Maxwell equations:

$$\operatorname{div} \vec{E} = 0 \quad , \quad \operatorname{div} \vec{B} = 0$$

$$\operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \underset{\uparrow}{=} i\omega \vec{B} \quad , \quad \operatorname{rot} \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \underset{\uparrow}{=} -\frac{i\omega}{c^2}\vec{E}$$

standing wave $\vec{B} \sim e^{-i\omega t}$ $\qquad$ $\vec{E} \sim e^{-i\omega t}$

- boundary conditions: