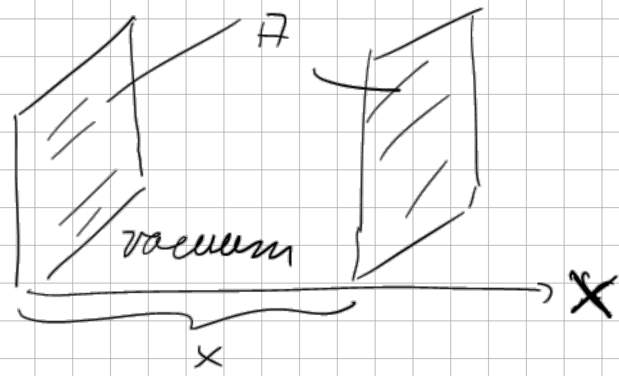
2.17 Casimir Effect:

attractive force between metal plates due to quantum fluctuations of the electromagnetic field



- Maxwell equations: see last time
- Boundary conditions (metal plates)

$$\operatorname{div}\vec{B} = 0 \quad \xrightarrow{\text{Gauß law}} \quad \vec{e}_x \cdot \vec{B} = 0$$

transversal

$$\operatorname{rot}\vec{E} = i\omega\vec{B} \quad \xrightarrow{\text{Stokes law}} \quad \vec{e}_x \times \vec{E} = \vec{0}$$

longitudinal

for $x = 0, d$

$$\vec{B} \sim e^{-i\omega t}$$

Two types of modes with $\vec{k}_\perp = k_y \vec{e}_y + k_z \vec{e}_z$

1) Transversal electric modes (TE): $\vec{e}_x \cdot \vec{E} = 0$

$$\vec{E}(x,y,z,t) = N\omega \sin(k_x x)\, \vec{e}_x \times \vec{k}_\perp\, e^{i(k_y y + k_z z - \omega t)}$$

$$\vec{B}(x,y,z,t) = N\left\{ k_\perp^2 \sin(k_x x)\,\vec{e}_x + i k_x \cos(k_x x)\,\vec{k}_\perp \right\} e^{i(k_y y + k_z z - \omega t)}$$

solve Maxwell equations

$\vec{e}_x \cdot \vec{B} = 0$ for $x = 0, d$: $\Rightarrow \sin(k_x x)\big|_{x=0} = 0 = \sin(k_x x)\big|_{x=d}$

$\Rightarrow k_x = \frac{\pi}{d} n; \; n = \cancel{0}, 1, 2, 3, \ldots$

$n = 0$: $\vec{E} = \vec{0}$ and, due to $k_x = 0$, also $\vec{B} = \vec{0}$ $\Rightarrow$ no mode

2) Transversal magnetic modes (TM): $\vec{e}_x \cdot \vec{B} = 0$

$$\vec{B}(x,y,z,t) = N\frac{\omega}{c^2}\cos(k_x x)\,\vec{k}_\perp \times \vec{e}_x \; e^{i(k_y y + k_z z - \omega t)}$$

$$\vec{E}(x,y,z,t) = N\left\{ k_T^2 \cos(k_x x)\,\vec{e}_x - i k_x \sin(k_x x)\,\vec{k}_\perp \right\} e^{i(k_y y + k_z z - \omega t)}$$

$\vec{e}_x \times \vec{E} = 0$ for $x = 0, d$ $\Rightarrow$ $\sin(k_x x)\big|_{x=0} = 0$ $\sin(k_x d)\big|_{x=d}$

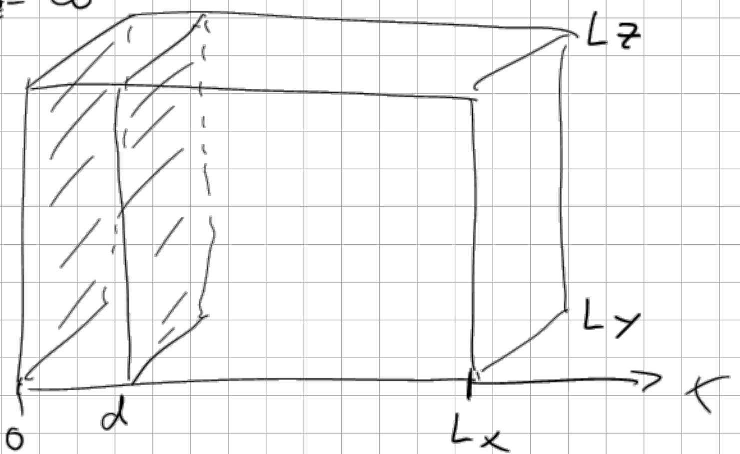
$\Rightarrow k_x = \frac{\pi}{d} n$; $n = 0, 1, 2, 3, \ldots \Rightarrow n = 0$ is here NOT excluded

In addition: dispersion

$$\omega = c\sqrt{k_x^2 + k_\perp^2} = c\sqrt{\left(\frac{\pi}{d}n\right)^2 + k_\perp^2} \quad \text{with} \begin{cases} n = 1, 2, 3, \ldots & TE \\ n = 0, 1, 2, 3, \ldots & TM \end{cases}$$

Vacuum energy inside the metal plates:

$$E_{\text{plates}}^{\text{inside}} = 2\sum_{n=0}^{\infty}\left(1 - \frac{1}{2}\delta_{n,0}\right)\sum_{n_y=-\infty}^{+\infty}\sum_{n_z=-\infty}^{+\infty}\frac{1}{2}\hbar c\sqrt{\left(\frac{\pi}{d}n\right)^2 + k_\perp^2}$$



periodic boundary conditions

$$e^{ik_y y}\Big|_{y=0} = e^{ik_y y}\Big|_{y=L_y}$$

$$\Rightarrow k_y = \frac{2\pi}{L_y} n_y \; ; \; n_y \in \mathbb{Z}$$

in principle k_y, k_z discrete

$$\sum_{n_y=-\infty}^{+\infty} \approx L_y \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi}$$

Euler McLaurin formula

$$1 = \Delta n_y = \frac{L_y}{2\pi} dk_y$$

$$E_{\text{plates}}^{\text{inside}} = 2\sum_{n=0}^{\infty}\left(1 - \frac{1}{2}\delta_{n,0}\right)\underbrace{L_y L_z}_{=A}\int\frac{d^2 k_\perp}{(2\pi)^2}\,\frac{1}{2}\hbar c\sqrt{\left(\frac{\pi}{d}\right)^2 n^2 + k_\perp^2}$$

$$E^{\text{outside}}_{\text{plates}} = \underbrace{2}_{\text{transversal degrees}} (\cancel{L_x} - d) \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} A \int \frac{d^2k_\perp}{(2\pi)^2} \frac{1}{2}\hbar c \sqrt{k_x^2 + k_\perp^2}$$

$$E^{\text{no plates}}_{\text{total}} = 2 \quad \cancel{L_x} \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} A \int \frac{d^2k_\perp}{(2\pi)^2} \frac{1}{2}\hbar c \sqrt{k_x^2 + k_\perp^2}$$

Change of vacuum due to installing 2 plates:

$$E_C = E^{\text{inside}}_{\text{plates}} + E^{\text{outside}}_{\text{plates}} - E^{\text{no plates}}_{\text{total}}$$

$$= \hbar c A \sum_{n=0}^{\infty} \left(1 - \frac{1}{2}\delta_{n,0}\right) \int \frac{d^2k_\perp}{(2\pi)^2} \sqrt{\frac{\pi^2}{d^2}n^2 + k_\perp^2}$$

$$- \hbar c A \frac{d}{\cancel{2}\pi} \underbrace{\int_{\cancel{-\infty}\,0}^{+\infty} dk_x}_{} \int \frac{d^2k_\perp}{(2\pi)^2} \sqrt{k_x^2 + k_\perp^2}$$

$$\underline{\underline{k_x = \frac{\pi}{d} n}} \qquad \int_0^\infty dn \qquad \sqrt{\left(\frac{\pi}{d}n\right)^2 + k_\perp^2}$$

$$E_c = \hbar c A \boxed{\Big(\underbrace{\sum_{n=0}^{\infty}}_{\text{infinite}}{}' - \underbrace{\int_0^\infty dn}_{\text{infinite}}\Big)} \int \frac{d^2k_\perp}{(2\pi)^2} \sqrt{\frac{\pi^2}{d^2}n^2 + k_\perp^2}$$ → finite Casimir energy

$$\boxed{\sum_{n=0}^{\infty}{}' f_n = \sum_{n=0}^{\infty} f_n - \frac{1}{2} f_0}$$

2.17.2 Analytic Continuation:

Regularization: Tame infinities by a regularization procedure where a new parameter is introduced in such a way that the infinity corresponds to a certain limit of that parameter

Problem sheet 3: ultraviolet cut off (UV) Λ

Here: Dimensional regularisation invented by Gerard 't Hooft and Martinus Veltman in 1972 (nobel prize!)

Instead of $D=3$ one considers a suitable dimension D where integrals converge and then, due to analytic continuation, you put $D=3$ at the end

$$E_C = \hbar c A \left(\sum_{n=0}^{\infty} - \int_0^{\infty} dn \right) \underbrace{\int \frac{d^{D-1} k_\perp}{(2\pi)^{D-1}} \sqrt{\left(\frac{\pi}{d} n\right)^2 + \vec{k}_\perp^{\,2}}}$$

UV cut off $\to$ superficial divergency:

$$\sim \int_0^{\Lambda} dk_\perp \, k_\perp^{D-2} \cdot k_\perp \sim \Lambda^D \xrightarrow{\Lambda \to \infty} \text{finite}$$

$\downarrow$

$D \leq 0$

Plan: Evaluate $\vec{k}_\perp$-integral with Schwinger trick: $\to$ Gauß

$$\frac{1}{a^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} d\tau \, \tau^{x-1} e^{-a\tau} \; ; x > 0 \; , \; a > 0$$

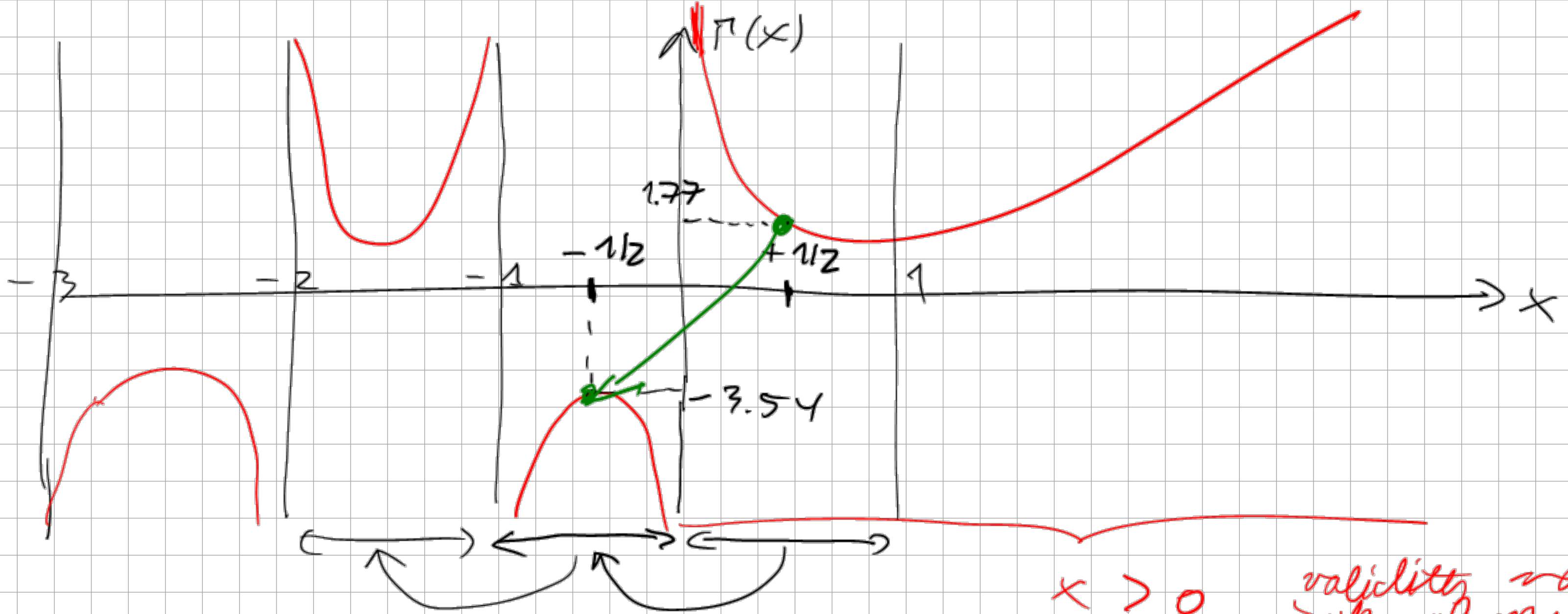
$$\sqrt{\left(\frac{\pi}{d} n\right)^2 + k_\perp^2} = \frac{1}{\left[\left(\frac{\pi}{d} n\right)^2 + k_\perp^2\right]^{-1/2}} \overset{x=-\frac{1}{2}}{=} \frac{1}{\Gamma(-\frac{1}{2})} \int_0^{\infty} d\tau \, \tau^{-\frac{3}{2}} e^{-\left(\frac{\pi^2}{d^2} n^2 + \vec{k}_\perp^{\,2}\right)}$$

advantage: $\vec{k}_\perp$-integrals are just Gauß function

disadvantage: $\Gamma(-\frac{1}{2}) = \; ? \Leftarrow$

Gamma function: $\Gamma(x) = \int_0^{\infty} d\tau \, \tau^{x-1} e^{-\tau} \; ; x > 0$

recursion formula: $\Gamma(x+1) = x \cdot \Gamma(x) \Rightarrow \Gamma(n+1) = n!$



x > 0

validity range of integral representation of Gamma function

analytic continuation via recursion formula

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

$x \in (-1,0) \longleftrightarrow x \in (0,1)$

$x \in (-2,-1) \longleftrightarrow x(-1,0)$

$$\Gamma\left(\tfrac{1}{2}\right) = \Gamma\left(-\tfrac{1}{2}+1\right) = -\tfrac{1}{2}\,\Gamma\left(-\tfrac{1}{2}\right)$$

$$\Rightarrow \Gamma\left(-\tfrac{1}{2}\right) = -2\cdot\sqrt{\pi} \approx -3.54$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty d\tau\,\tau^{\frac{1}{2}-1}e^{-\tau} \overset{\tau=u^2}{\underset{\text{Gauß}}{=}} \sqrt{\pi} \approx 1.77$$

$$\int \frac{d^{D-1}k_\perp}{(2\pi)^{D-1}}\sqrt{\frac{\pi^2 n^2}{d^2}+\vec{k}_\perp^{\,2}} = \frac{1}{-2\sqrt{\pi}}\int_0^\infty d\tau\,\tau^{-\frac{3}{2}}\,e^{-\frac{\pi^2 n^2}{d^2}\tau^2}\underbrace{\int \frac{d^{D-1}k_\perp}{(2\pi)^{D-2}}\,e^{-\vec{k}_\perp^{\,2}\tau}}_{\overset{\text{Gauß}}{=}\ \frac{1}{(4\pi\ 2)^{\frac{D-1}{2}}}\ \tau^{-\frac{D+1}{2}}}$$

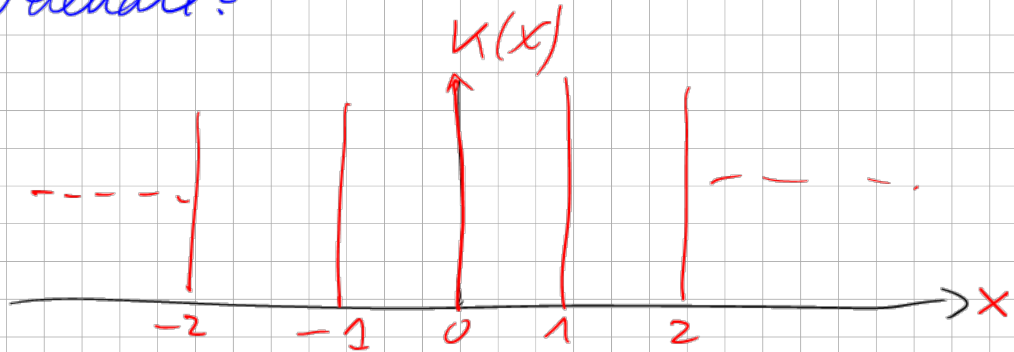
Schwinger integral

intermediate result:

$$E_C = \frac{-\hbar c A}{2\sqrt{\pi}\,(4\pi)^{\frac{D-1}{2}}}\int_0^\infty d\tau\,\tau^{-\frac{D}{2}-1}\underbrace{\left(\sum_{n=0}^{\infty} - \int_0^\infty dn\right)}\,e^{-\frac{\pi^2}{d^2}n^2\tau^2}$$

how to evaluate?

2.17.3 Poisson Sum Formula:



comb function:

$$K(x) = \sum_{n=-\infty}^{+\infty} \delta(x-n)$$

periodic: $K(x) = K(x+k)$; $k \in \mathbb{Z}$ $\Rightarrow$ Fourier series

$$K(x) = \sum_{m=-\infty}^{+\infty} K_m \, e^{-2\pi i m x} \; ; \quad K_m = \int_{-1/2}^{+1/2} dx \; K(x)\, e^{2\pi i m x} = 1$$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} \delta(x-n) = \sum_{m=-\infty}^{+\infty} e^{-2\pi i m x}$$ distributional identity $\Big| \int_{-\infty}^{+\infty} dx \; f(x)$

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx \; f(x)\, e^{-2\pi i m x}$$

mathematical relation between two sums

Specialization: $f(x) = f(-x)$ even

$$\underbrace{\cancel{2}\sum_{n=0}^{\infty} f(n) - \frac{1}{2} f(0)}_{\sum_{n=0}^{\infty}{}' f(n)} = \underbrace{\cancel{2}\int_{0}^{+\infty} dx \; f(x)}_{m=0} + \cancel{2}\sum_{m=1}^{\infty} \int_{-\infty}^{+\infty} dx \; f(x) \cos(2\pi m x)$$

SUM

Integral

$$\sum_{n=0}^{\infty}{}' f(n) - \int_0^{\infty} dn \; f(n) = \mathrm{Re} \sum_{m=1}^{\infty} \int_{-\infty}^{+\infty} dx \; f(x)\, e^{2\pi i m x}$$

Poisson sum formula for even functions

$$\left(\sum_{n=0}^{\infty} - \int_0^\infty dn\right) e^{-\frac{\pi^2 n^2 \tau^2}{d^2}} = \cancel{\mathrm{Re}} \sum_{m=1}^{\infty} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-\frac{\pi^2 x^2 \tau^2}{d^2}} e^{2\pi i m x}}_{\sqrt{\frac{d}{\pi \tau}}\, e^{-\frac{d^2 m^2}{\tau}}}$$

$$E_c = -\frac{\hbar c A}{2\sqrt{\pi}\,(4\pi)^{\frac{D-1}{2}}} \cdot \frac{d}{\sqrt{\pi}} \sum_{m=1}^{\infty} \underbrace{\int_0^\infty d\tau\, \tau^{-\frac{D}{2}-\frac{3}{2}} e^{-\frac{d^2 m^2}{\tau}}}_{\tau = \frac{1}{u}}$$

$$= \ldots = \frac{\Gamma\left(\frac{D+1}{2}\right)}{\left[d^2 m^2\right]^{\frac{D+1}{2}}}$$

NOW: analytic continuation of result to D=3:

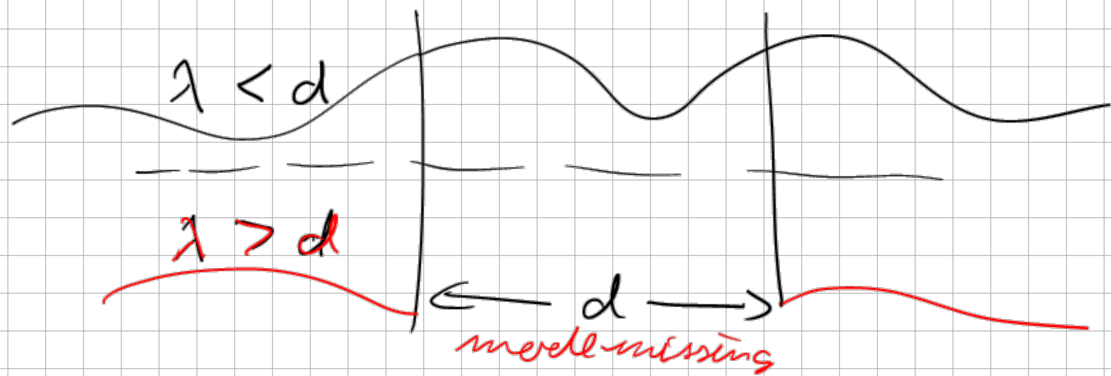
$$E_c = -\frac{\hbar c A\, \Gamma\left(\frac{D+1}{2}\right)}{2\pi\,(4\pi)^{\frac{D-1}{2}}} \underbrace{\sum_{m=1}^{\infty} \frac{1}{m^{D+1}}}_{= \zeta(D+1) \text{ Riemann zeta function}} \overset{D=3}{=} -\frac{\pi^2}{720} \frac{\hbar c A}{d^3}$$

natural constants

geometric quantities

$\zeta(4) = \frac{\pi^4}{90}$ → Appendix B

2.17.4 Physical discussion:



minus sign: attractive

higher mode density outside than inside → plates are pressed together

Casimir force: $F_c = -\frac{\partial E_c}{\partial d} = -\frac{\pi^2}{240} \frac{\hbar c A}{d^4}$

relevant at distances below 1 μm → relevant for nanotechnology

$d = 1\,\mu m$, $A = 1\,(\mu m)^2$ $\Rightarrow$ $F_c = 1.3\,pN$ $(1p = 10^{-12})$

$\Rightarrow$ measurable with force microscope