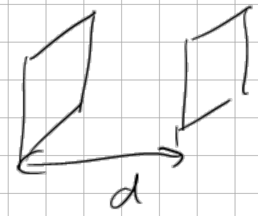
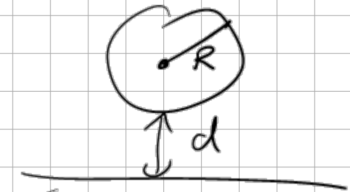
Physical discussion around Casimir Effect:

$$E_c = -\frac{\pi^2}{720}\frac{\hbar c A}{d^3}, \qquad F_c = -\frac{\partial E_c}{\partial d} = -\frac{\pi^2 \hbar c A}{240\, d^4}$$



experimental problem: plane-parallel plates

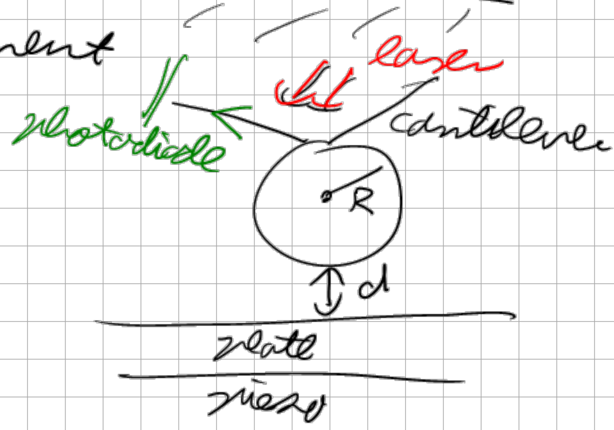
instead: plate + sphere



Casimir force: $F_c = -\frac{\pi^3 \hbar c R}{360\, d^3}$, $R \gg d$

PRL 81, 4549 (1998) → high-precision measurement

$R = 98\,\mu m$, $d = 0.1 - - - 0.9\,\mu m$



Application: search for non-Newtonian gravity in μ m range

- NJP 8, 237 (2006)
- Class. and Quantum Grav. 32, 033001 (2015)

possibility of "fifth" force

$$\frac{1}{r} \to \frac{1}{r}\left\{1 + \underbrace{A}_{\text{strength}}\, e^{-r/\lambda}\right\}$$ range

Precise measurements need theoretical corrections:

- vary geometry: two spheres
- vacuum and thermal fluctuations
- surfaces may not be perfectly conducting: material sciences enters

- surface roughnesses
- magnetic materials: other boundary conditions

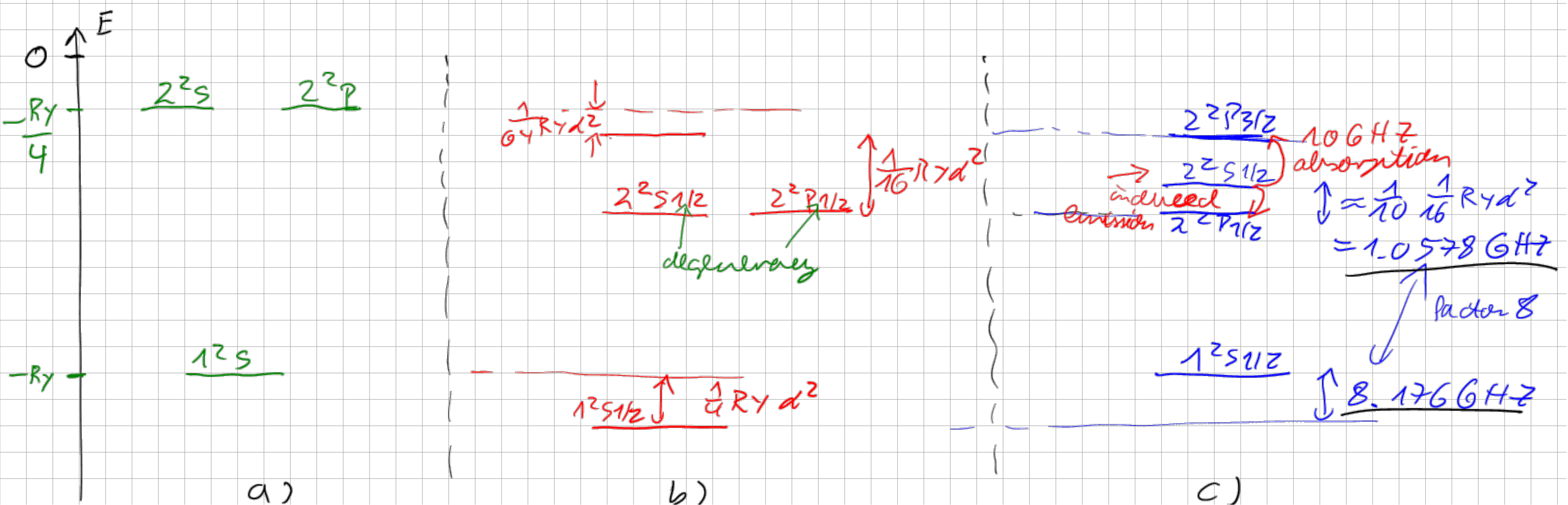
2.18 Lamb Shift:

- Experiment of Willis Lamb from 1947:

degeneracy of $2^2S_{1/2}$ and $2^2P_{1/2}$ level according to Dirac theory gets lifted due to vacuum fluctuations of electromagnetic field

- Lamb shift played important role in QED + atomic physics
- Possible student talk: Scully et al. Virtual photons: From the Lamb shift to black holes, Opt. Photonics News 29, 34 (2018)

2.18.1 Energy Levels of Hydrogen Atom:



a) Schrödinger theory:

$$E_n = -\underbrace{Ry}_{13.6\,eV}\,\frac{1}{n^2}$$

$$Ry = \frac{1}{2} M c^2 \alpha^2 = \frac{1}{2}\,(0.511\,MeV)\left(\frac{1}{137}\right)^2 = 13.6\,eV$$ Rydberg energy

Sommerfeld fine structure constant: $\alpha = \frac{e^2}{4\pi\hbar\varepsilon_0 c} \approx \frac{1}{137}$

notation: $n^{(2S+1)}\underbrace{\ell}$

$(2S+1) \to$ spin multiplicity

n: principal quantum number

ℓ: orbital quantum number

$\ell = 0, 1, 2, 3 \ldots$
$\quad\; s\;\; p\;\; d\;\; f \ldots$

b) Dirac theory (Quantum Mechanics II - next term): with spin, relativistic

$$E_{nj} = Mc^2\sqrt{1 - \frac{\alpha^2}{n^2 + 2(n-j-\frac{1}{2})\left[(j+\frac{1}{2})^2 - \alpha^2\right] - j - 1/2}}$$

$$\approx Mc^2 + E_n + E^{FS}_{nj} + \ldots$$

E^{FS} ← fine structure

Taylor expansion in α^2

$$E^{FS}_{nj} = -\frac{Ry\,\alpha^2}{n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right), \quad \underline{j = |\ell \pm \tfrac{1}{2}|}$$ total angular momentum

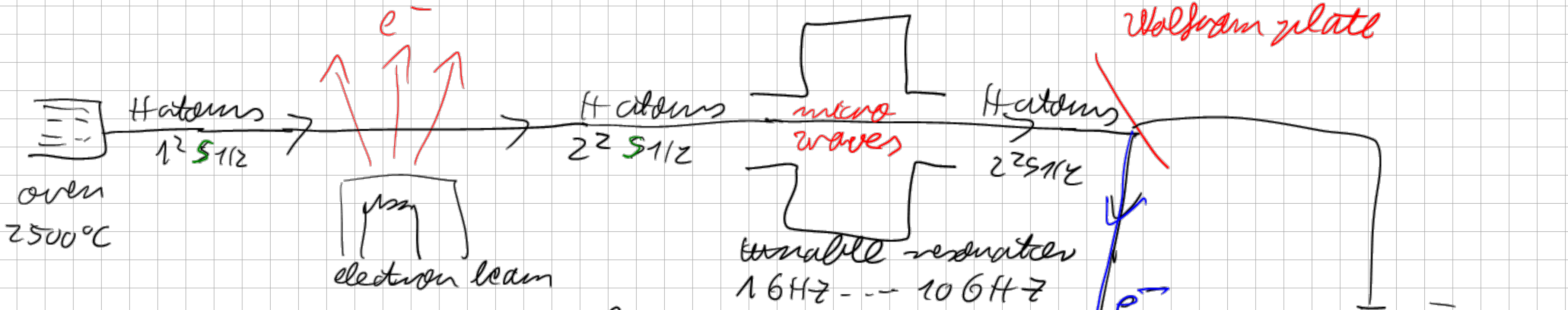
- Sommerfeld: Kepler ellipses instead of Bohr radii
- – : large velocity → larger mass → larger Ry
- E^{FS}_{nj}:
  > relativistic energy - momentum dispersion
  > spin - orbit coupling

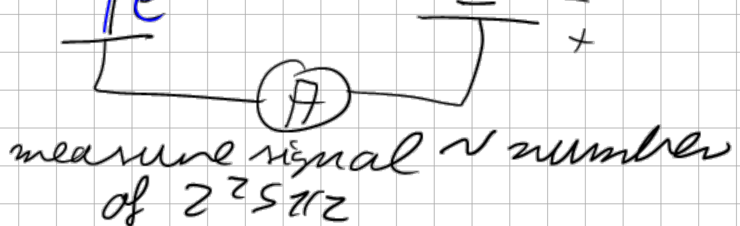
→ Darwin term: Zitterbewegung of electron due to electron-positron creation/annihilation in vacuum → only s electrons

notation: $n^{2S+1} l_j \to j = |l \pm \frac{1}{2}|$

2.18.2 Detection in Microwave Range:

Set up of Lamb and Retherford 1947





- Oven dissociation of H_2 molecules
- Detection in Optical Range:
  student talk: Zalen/Wolf, Demtröder

2.18.3 Qualitative Explanation:

- Heisenberg uncertainty between energy and time

- Infinite amount of energy for both electron in an atom and for a freely moving atom

- But: experimentally atoms get a finite energy shift
- Highest accuracy between theory and experiment in all sciences

2.18.4 Vacuum Fluctuations:

- Lamb shift stems from QED
- But: Lamb shift is leading correction
  - matter can still be treated non-relativistically
  - perturbation theory for Schrödinger theory
- Starting point: Coulomb potential: $V(\vec{x}) = -\frac{e^2}{4\pi\varepsilon_0 |\vec{x}|}$
- change due to fluctuations of electromagnetic field

$$\underset{\text{steady state}}{V(\vec{x}} + \underset{\text{fluctuation}}{\vec{s})} \overset{|\vec{s}| \ll |\vec{x}|}{=\!=} V(\vec{x}) + \sum_{i=1}^{3} \frac{\partial V(\vec{x})}{\partial x_i} s_i + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^2 V(\vec{x})}{\partial x_i \partial x_j} s_i s_j + \ldots$$

- Average $\langle \cdot \rangle$ over fluctuations: $\langle s_i \rangle = 0$, $\langle s_i s_j \rangle = \frac{1}{3} \underline{\delta_{ij}} \langle \vec{s}^2 \rangle$

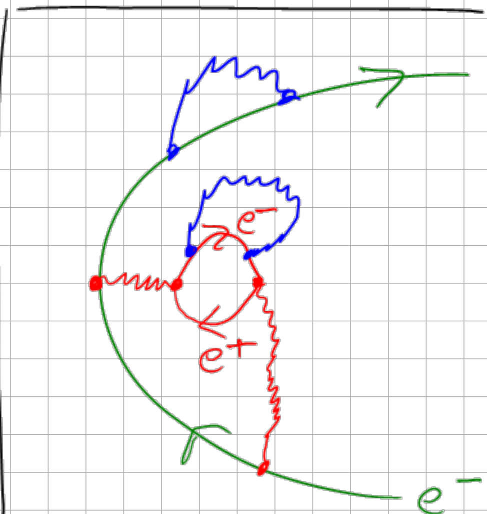
$$\langle V(\vec{x}+\vec{s}) \rangle - V(\vec{x}) = \underline{\frac{1}{6} \Delta V(\vec{x}) \langle \vec{s}^2 \rangle} + \ldots$$

$$\Delta E_L \underset{\text{1st order perturbation theory}}{=} \int d^3x \, \psi^*_{nlm}(\vec{x}) \underline{\left[\langle V(\vec{x}+\vec{s}) \rangle - V(\vec{x}) \right]} \psi_{nlm}(\vec{x})$$

$$= \frac{1}{6} \frac{-e^2}{4\pi\varepsilon_0} \underbrace{\Delta \frac{1}{|\vec{x}|}}_{= -4\pi \delta(\vec{x})}$$

$$= \underbrace{+}_{\text{positive}} \frac{e^2}{6\varepsilon_0} |\psi_{nlm}(\vec{0})|^2 \langle \vec{s}^2 \rangle$$

only s-states are involved

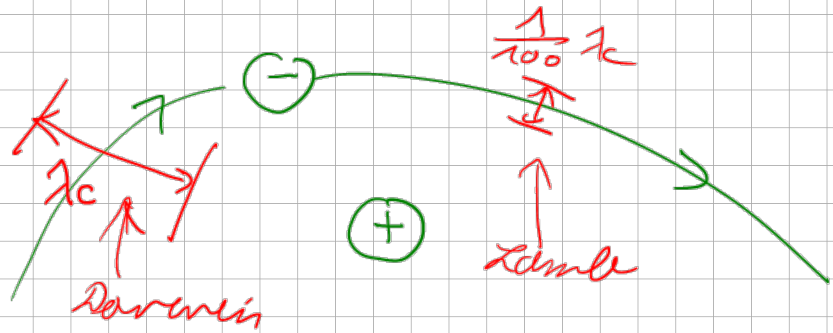


→ same structure for Lamb shift and Darwin term

Darwin term
virtual electron-positron pairs

$\sqrt{\langle \vec{s}^2 \rangle} = \lambda_c = 2\pi \frac{\hbar}{mc}$

Compton wave length



Lamb shift
virtual photons

$\langle \vec{s}^2 \rangle \approx \frac{1}{100} \lambda_c$

- $|\Psi_{n00}(\vec{0})|^2 = \frac{1}{\pi a_B^3 n^3}$ / a_B : Bohr radius

$$\Delta E_L = \frac{e^2}{6\pi\varepsilon_0 a_B^2 n^3} \underbrace{\langle \vec{s}^2 \rangle}_{\text{only weakly dependent on } n} \rightarrow \text{neglect this}$$

$$\frac{\Delta E_L(2^2 S_{1/2})}{\Delta E_L(1^2 S_{1/2})} = 8 \mathrel{\hat{=}} \frac{8.176\,\mathrm{GHz}}{1.06\,\mathrm{GHz}} = 7.7$$

- Calculation of $\langle \vec{s}^2 \rangle$:

↙ classical → quantum mechanical approach

→ same result but it is divergent