4.5.5 Dressed States:

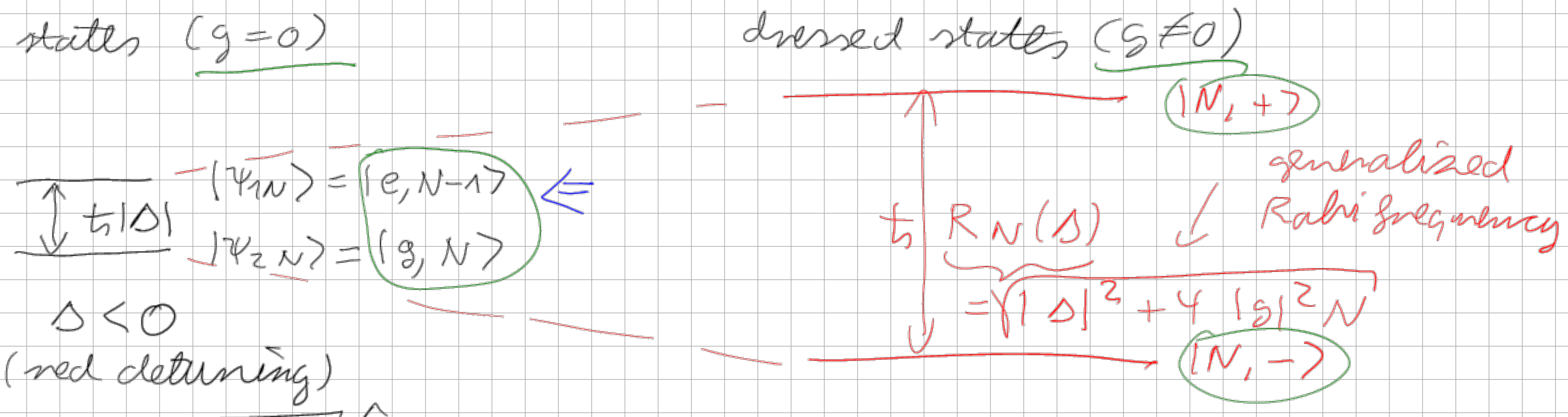
Different methods to determine Jaynes–Cummings dynamics

- so far: direct solution of Schrödinger equation, no detuning ($\Delta = 0$)
- Now: solution for $\Delta \neq 0$, based on stationary states of Jaynes–Cummings Hamiltonian

$$\hat{H}_{JC} = \underbrace{\hbar\omega \hat{a}^{+}\hat{a} + \hbar\omega_0 \sigma_+ \sigma_-}_{= \hat{H}^{(0)}} + \underbrace{\hbar g \hat{a} \sigma_+ + \hbar g^{*} \hat{a}^{+} \sigma_-}_{= \hat{H}^{(int)}} \; ; \; g = |g| e^{i\varphi_g}$$

(rotating-wave approximation)

$\rightarrow$ eigenstates of $\hat{H}_{JC}$



$$\hat{H}_{JC} |N,\pm\rangle = E_{\pm,N} |N,\pm\rangle$$

$$= \hbar\omega N - \frac{\hbar}{2}\Delta \pm \frac{\hbar}{2} R_N(\Delta)$$

$$\begin{pmatrix} |N,+\rangle \\ |N,-\rangle \end{pmatrix} = U(\Delta, g) \begin{pmatrix} |\psi_{1N}\rangle \\ |\psi_{2N}\rangle \end{pmatrix} \qquad U(\Delta,g) = \begin{pmatrix} \cos\left(\frac{\phi_N}{2}\right) & \sin\left(\frac{\phi_N}{2}\right)e^{-i\varphi_g} \\ -\sin\left(\frac{\phi_N}{2}\right)e^{i\varphi_g} & \cos\left(\frac{\phi_N}{2}\right) \end{pmatrix}$$

dressed states $(g \neq 0)$ bare states $(g=0)$ $\phi_N = \arctan \frac{2|g|N}{-\Delta}$

1) special case: $U(\Delta, g=0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2) special case: $\Delta = 0$

- bare states: degenerate
- splitting of dressed states: $R_N(0) = 2|g|N$
- angle $\phi_N = \pi/2$

$$\Rightarrow U(\Delta=0, g) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{-i\varphi_g} \\ e^{i\varphi_g} & 1 \end{pmatrix}$$

Initial condition:

- atomic state: $|e\rangle$
- field state: any photon distribution

$$|\psi(0)\rangle = |e\rangle \sum_{N=0}^{\infty} c_N |N\rangle \underset{N'=N+1}{=} |e\rangle \sum_{N'=1}^{\infty} c_{N'-1} |N'-1\rangle = \sum_{N'=1}^{\infty} c_{N'-1} |\psi_{1N'}\rangle$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_{JC} t} |\psi(0)\rangle$$

$$\begin{pmatrix} |N,+\rangle \\ |N,-\rangle \end{pmatrix} = U(\Delta, g) \begin{pmatrix} |\psi_{1N}\rangle \\ |\psi_{2N}\rangle \end{pmatrix} \Leftrightarrow \begin{pmatrix} |\psi_{1N}\rangle \\ |\psi_{2N}\rangle \end{pmatrix} = U^{+}(\Delta, g) \begin{pmatrix} |N,+\rangle \\ |N,-\rangle \end{pmatrix}$$

$$= \cos\left(\frac{\phi_N}{2}\right) |N,+\rangle - \sin\left(\frac{\phi_N}{2}\right) e^{-i\varphi_g} |N,-\rangle$$

$$e^{-\frac{i}{\hbar}\hat{H}_{JC}t}|N,\pm\rangle = e^{-\frac{i}{\hbar}E_{N,\pm}t}|N,\pm\rangle$$

$$|\Psi(t)\rangle = \sum_{N=1}^{\infty} c_{N-1}\left\{\cos\left(\frac{\theta_N}{2}\right)e^{-\frac{i}{\hbar}E_{N,+}t}\underbrace{|N,+\rangle} - \sin\left(\frac{\theta_N}{2}\right)e^{-i\varphi_g}e^{-\frac{i}{\hbar}E_{N-}t}\underbrace{|N-\rangle}\right\}$$

rewrite dressed states back into bare states

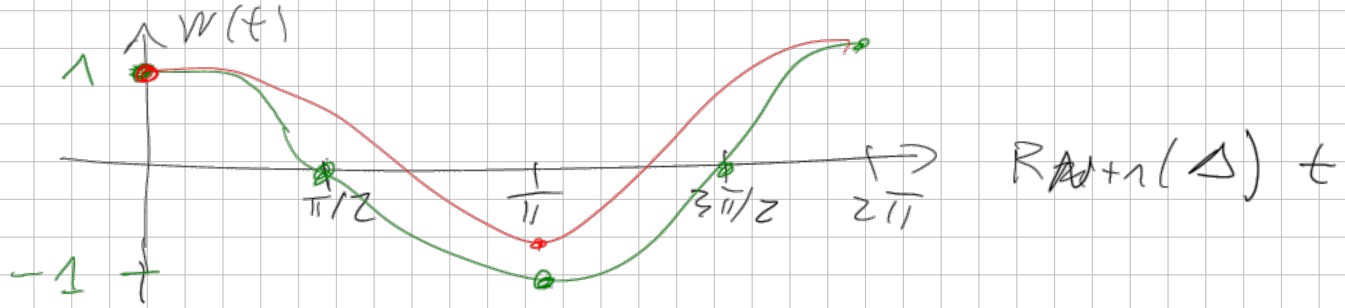
$$= \dots = \underbrace{|\Psi_e(t)\rangle}|e\rangle + \underbrace{|\Psi_g(t)\rangle}|g\rangle$$

photonic amplitudes

$$W(t) = \langle\Psi_e(t)|\Psi_e(t)\rangle - \langle\Psi_g(t)|\Psi_g(t)\rangle$$

$$= \dots = \sum_{N=0}^{\infty}|c_N|^2\left\{\frac{\Delta^2}{\Delta^2+4|g|^2(N+1)} + \frac{4|g|^2(N+1)}{\Delta^2+4|g|^2(N+1)}\cos\left(\sqrt{\Delta^2+4|g|^2(N+1)}\,t\right)\right\}$$

- $\Delta = 0$: previous result ✓
- initial Fock state: one term is non-zero



$\Delta = 0$ $\Delta > 0$

- Δ larger: larger Rabi frequency + smaller amplitude
- $\Delta \to \infty$: $W(t) = \sum_{N=0}^{\infty}|c_N|^2 = 1$

no transition at all, system remains in excited state

4.5.6 Density-Matrix Approach:

- so far: $|\Psi(0)\rangle = |\Psi_{atom}\rangle \cdot |\Psi_{field}\rangle$ pure state
- Now: initially mixed state of atom and field
- Simplification: resonance, $\Delta = 0$

$$\hat H_{JC} = \underbrace{\hbar\omega(\hat a^+ \hat a + \sigma_+ \sigma_-)}_{=\hat H^{(0)}\ \text{unperturbed}} + \underbrace{\hbar g \hat a \sigma_+ + \hbar g^* \hat a^+ \sigma_-}_{=\hat H^{(int)}\ \text{perturbed}}$$

→ applying perturbation theory

→ still exact calculation possible

Evolution of density matrix:

Von Neumann equation: $i\hbar \frac{\partial}{\partial t}\hat\rho(t) = [\hat H, \hat\rho(t)]_-$

$$\Rightarrow \hat\rho(t) = e^{-\frac{i}{\hbar}\hat H t}\,\hat\rho(0)\,e^{+\frac{i}{\hbar}\hat H t}$$

Perturbation theory: Dirac interaction picture

- State vector: time dependent with $\hat H^{(0)}$

$$|\Psi_D(t)\rangle = e^{+\frac{i}{\hbar}\hat H^{(0)} t}\,|\Psi(t)\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\Psi_D(t)\rangle = \boxed{\hat H_D^{(int)}\cancel{(t)}}\,|\Psi_D(t)\rangle \Rightarrow |\Psi_D(t)\rangle = \hat U_D(t)\,\underbrace{|\Psi_D(0)\rangle}_{=|\Psi_S(0)\rangle}$$

$$\hat U_D(t) = e^{-\frac{i}{\hbar}\hat H_{JC}^{(int)} t}$$

$$\hat O_D(t) = e^{+\frac{i}{\hbar}\hat H^{(0)} t}\,\hat O_S\,e^{-\frac{i}{\hbar}\hat H^{(0)} t}$$

a) Photon operators:

$$\hat a_D(t) = e^{\frac{i}{\hbar}\hat H^{(0)} t}\,\hat a\, e^{-\frac{i}{\hbar}\hat H^{(0)} t} = e^{i\omega \hat a^+\hat a t}\,\hat a\, e^{-i\omega \hat a^+ \hat a t}$$

$$i\hbar \frac{\partial}{\partial t}\hat a_D(t) = \dots = \hbar\omega\, \hat a_D(t) \Rightarrow \hat a_D(t) = e^{-i\omega t}\hat a, \quad \hat a_D^+(t) = e^{i\omega t}\hat a^+$$

$$[\hat a_D(t), \hat a_D^+(t)]_- = 1 = [\hat a, \hat a^+]_-$$

b) Atomic operators

$$\sigma_{+D}(t) = e^{i\omega t}\sigma_+, \quad \sigma_{-D}(t) = e^{-i\omega t}\sigma_-$$

c)

$$\hat H^{(int)}_{JC\,D}(t) = e^{\frac{i}{\hbar}\hat H^{(0)}_{JC}t}\,\underbrace{\hat H^{(int)}_{JC}}\, e^{-\frac{i}{\hbar}\hat H^{(0)}_{JC}t}$$

$$= \hbar g\, \hat a \underset{=1}{\wedge} \sigma_+ + \hbar g^* \hat a^+ \underset{=1}{\wedge} \sigma_-$$

$$= \hbar g\, \hat a_D(t)\sigma_{+D}(t) + \hbar g^* \hat a^+_D(t)\sigma_{-D}(t)$$

$$= \hbar g\, \cancel{e^{-i\omega t}}\,\hat a\, \cancel{e^{+i\omega t}}\,\sigma_+ + \hbar g^*\, \cancel{e^{i\omega t}}\,\hat a^+\, \cancel{e^{-i\omega t}}\,\sigma_- \equiv \hat H^{(int)}_{JC}$$

time independent due to resonance

$\Delta = 0$

$$\Rightarrow \hat U_{DJC}(t) = e^{-\frac{i}{\hbar}\hat H^{(I)}_{JC}t} = e^{-i(g\hat a\sigma_+ + g^*\hat a^+\sigma_-)t}$$

Fock states of photons

2 level system

→ exponential of operators

$$\hat U_{DJC}(t) = e^{-i\hat O t}, \quad \hat O = g\hat a\sigma_+ + g^*\hat a^+\sigma_- = \begin{pmatrix}0 & g\hat a\\ g^*\hat a^+ & 0\end{pmatrix}$$

Taylor series: odd and even powers of $\hat{O}$

$$\hat{U}_{DJC}(t) = \hat{\sigma}_0 + \sum_{n=1}^{\infty} \frac{(-it)^{2n}}{(2n)!} \hat{O}^{2n} + \sum_{n=0}^{\infty} \frac{(-it)^{2n+1}}{(2n+1)} \hat{O}^{2n+1}$$

$$\hat{O}^2 = \begin{pmatrix} 0 & g\hat{a} \\ g^*\hat{a}^+ & 0 \end{pmatrix} \begin{pmatrix} 0 & g\hat{a} \\ g^*\hat{a}^+ & 0 \end{pmatrix} = |g|^2 \begin{pmatrix} \hat{a}\hat{a}^+ & 0 \\ 0 & \hat{a}^+\hat{a} \end{pmatrix}$$

$$\hat{O}^3 = \ldots = |g|^2 \begin{pmatrix} 0 & g\hat{a}\hat{a}^+\hat{a} \\ g^*\hat{a}^+\hat{a}\hat{a} & 0 \end{pmatrix}$$

$$\hat{O}^4 = \hat{O}^2 \hat{O}^2 = \ldots = |g|^4 \begin{pmatrix} (\hat{a}\hat{a}^+)^2 & 0 \\ 0 & (\hat{a}^+\hat{a})^2 \end{pmatrix}$$

$$\hat{O}^{2n} = |g|^{2n} \begin{pmatrix} (\hat{a}\hat{a}^+)^n & 0 \\ 0 & (\hat{a}^+\hat{a})^n \end{pmatrix}, \quad \hat{O}^{2n+1} = |g|^{2n} \begin{pmatrix} 0 & g\hat{a}(\hat{a}^+\hat{a})^n \\ g^*\hat{a}^+(\hat{a}\hat{a}^+)^n & 0 \end{pmatrix}$$

→ complete induction: left for you to prove

even orders:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{(2n)!}$$

odd orders:

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\hat{U}_{DJC}(t) = \begin{pmatrix} \hat{C}(t) & \hat{S}'(t) \\ \hat{S}(t) & \hat{C}'(t) \end{pmatrix}, \quad \hat{U}^+_{DJC}(t) = \begin{pmatrix} \hat{C}(t) & -\hat{S}'(t) \\ -\hat{S}(t) & \hat{C}'(t) \end{pmatrix}$$

$$\hat{C}(t) = \cos\left(|g|t\sqrt{\hat{a}\hat{a}^+}\right), \quad \hat{C}'(t) = \cos\left(|g|t\sqrt{\hat{a}^+\hat{a}}\right)$$

$$\hat{S}(t) = -i\sqrt{\frac{g^*}{g}}\,\hat{a}\,\frac{\sin\left(|g|t\sqrt{\hat{a}\hat{a}^+}\right)}{\sqrt{\hat{a}\hat{a}^+}}, \quad \hat{S}'(t) = -i\sqrt{\frac{g}{g^*}}\,\frac{\sin\left(|g|t\sqrt{\hat{a}^+\hat{a}}\right)}{\sqrt{\hat{a}^+\hat{a}}}$$

$$\hat{S}_D(t) = \begin{pmatrix} \hat{C}(t) & \hat{S}'(t) \\ \hat{S}(t) & \hat{C}'(t) \end{pmatrix} \hat{S}(0) \begin{pmatrix} \hat{C}(t) & -\hat{S}'(t) \\ -\hat{S}(t) & \hat{C}'(t) \end{pmatrix}$$

Initial density matrix: $\hat{\rho}(0) = \underbrace{\hat{\rho}_A}_{\text{atom}} \otimes \underbrace{\hat{\rho}_F}_{\text{field}}$

both could be mixed

$|\psi_A\rangle = c_e|e\rangle + c_g|g\rangle$, $\langle\psi_A| = c_e^*\langle e| + c_g^*\langle g|$

$$\Rightarrow \hat{\rho}_A = |\psi_A\rangle\langle\psi_A| = \begin{pmatrix} c_e c_e^* & c_g^* c_e \\ c_g c_e^* & c_g c_g^* \end{pmatrix}$$

$|\psi_F\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$, $\langle\psi_F| = \sum_{n=0}^{\infty} c_n^* \langle n|$

$$\Rightarrow \hat{\rho}_F = |\psi_F\rangle\langle\psi_F| = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} c_{n_1} c_{n_2}^* |n_1\rangle\langle n_2|$$

$$\hat{\rho}_D(t) \begin{pmatrix} \hat{C}(t) & \hat{S}'(t) \\ \hat{S}(t) & \hat{C}'(t) \end{pmatrix} \hat{\rho}_F \begin{pmatrix} c_e c_e^* & c_e c_g^* \\ c_g c_e^* & c_g c_g^* \end{pmatrix} \begin{pmatrix} \hat{C}(t) & -\hat{S}'(t) \\ -\hat{S}(t) & \hat{C}'(t) \end{pmatrix}$$

$\Rightarrow$ multiply out

Atomic reduced density matrix:

$$\hat{\rho}_{DA}(t) = \mathrm{Tr}_F\left[\hat{\rho}_D(t)\right] = \sum_{n=0}^{\infty} \langle n|\hat{\rho}_D(t)|n\rangle = \cancel{\begin{pmatrix} \rho_{ee}^{DA}(t) & \rho_{eg}^{DA}(t) \\ \rho_{ge}^{DA}(t) & \rho_{gg}^{DA}(t) \end{pmatrix}}$$

normalization condition:

$$1 = \mathrm{Tr}_A\left[\hat{\rho}_{DA}(t)\right] = \rho_{ee}^{DA}(t) \; \rho_{gg}^{DA}(t) \Rightarrow \rho_{gg}^{DA}(t) = 1 - \rho_{ee}^{DA}(t)$$

atomic population inversion:

$$W(t) = \rho_{ee}^{DA}(t) - \rho_{gg}^{DA}(t) = 2\boxed{\rho_{ee}^{DA}(t)} - 1$$

only one term needed

$$\hat{\varrho}^{D}_{ee}(t) = c_e c_e^* \underline{\hat{C}(t)}\, \hat{\varrho}_F\, \hat{C}(t) - c_e c_g^* \hat{C}(t)\, \hat{\varrho}_F \hat{S}(t)$$
$$+ c_g c_e^* \underline{\hat{S}'(t)}\, \underbrace{\hat{\varrho}_F}\, \hat{C}(t) - c_g c_g^* \hat{S}'(t)\, \hat{\varrho}_F \underline{\hat{S}(t)}$$

critical photonic density matrix

Case 1: $c_e = 1$, $c_g = 0$

$$W(t) = \sum_{n=0}^{\infty} |c_n|^2 \cos\left(2|g|t\boxed{\sqrt{n+1}}\right)$$

$\Rightarrow$ previous result ✓ for vacuum Rabi oscillations + collapse / revival dynamics

Case 2: $c_e = 0$, $c_g = 1$

$$W(t) = \ldots = -\sum_{n=0}^{\infty} |c_n|^2 \cos\left(2|g|t\boxed{\sqrt{n}}\right) \Rightarrow \text{no vacuum oscillations}$$

Result: Necessary condition for having vacuum Rabi oscillations is $c_e = 1$

4.5.7 Large Detuning: dispersion interaction

- So far: JC model for $\Delta = 0$ and Δ small
  $\rightarrow$ atomic transition ("collapse and revival")
- Now: $\Delta \rightarrow \infty$
  - no atomic transition
  - dispersion interaction between atom and cavity field
    $\hat{=}$ bare states remain eigenstates but are shifted in energy

- Method: perturbation theory, but truncation at finite order is necessary