# 4.5.7 Large Detuning:

$$\hat H_{JC} = \underbrace{\hat H^{(0)}_{JC}} + \hat H^{(P)}_{JC} = \hbar g \hat a \sigma_+ + \hbar g^* \hat a^+ \sigma_-$$

$$= \hbar \omega \hat a^+ \hat a + \hbar \omega_0 \sigma_+ \sigma_-$$

Dirac interaction picture:

$$\hat a_D(t) = e^{\frac{i}{\hbar}\hat H^{(0)}_{JC} t}\, \hat a\, e^{-\frac{i}{\hbar}\hat H^{(0)}_{JC} t} = e^{-i\omega t}\hat a, \quad \hat a^+_D(t) = e^{i\omega t}\hat a^+$$

$$\sigma_{+D}(t) = e^{\frac{i}{\hbar}\hat H^{(0)}_{JC} t}\, \sigma_+\, e^{-\frac{i}{\hbar}\hat H^{(0)}_{JC} t} = e^{i\omega_0 t}\sigma_+, \quad \sigma_{-D}(t) = e^{-i\omega_0 t}\sigma_-$$

Explicit time dependence:

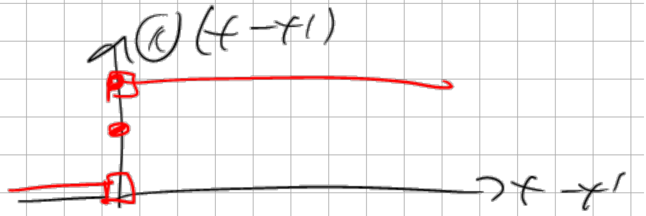
$$\hat H^{(P)}_{JCD}(t) = \hbar g\, \hat a_D \sigma_+(t) + \hbar g^* \hat a^+_D(t)\, \sigma_-(t) = \hbar g \boxed{e^{-i\Delta t}}\, \hat a \sigma_+ + \hbar g^* \boxed{e^{+i\Delta t}}\, \hat a^+ \sigma_-$$

$$i\hbar \frac{\partial}{\partial t}\hat U_D(t) = \hat H^{(P)}_{JCD}(t)\, \hat U_D(t), \quad \hat U_D(0) = 1$$

$$\Rightarrow \hat U_D(t) = \underbrace{\hat T}_{\text{time-ordering operator}} \left\{ \exp\left[ -\frac{i}{\hbar}\int_0^t dt'\, \hat H^{(P)}_{DJC}(t') \right] \right\}$$

$$\hat T\left[\hat A(t)\hat B(t')\right] = \underbrace{\Theta(t-t')}\, \hat A(t)\hat B(t') + \underbrace{\Theta(t'-t)}\, \hat B(t')\hat A(t)$$

Heaviside function




Taylor expansion:

$$\hat U_D(t) = 1 - \frac{i}{\hbar}\int_0^t dt'\, \hat H^{(P)}_{DJC}(t')\left[1 - \frac{i}{\hbar}\int_0^{t'} dt''\, \hat H^{(P)}_{JCD}(t'')\right] + \dots$$

insertion yields with rotating wave approximation due to large detuning $\Delta$

$$\int_0^t dt'\, e^{\pm i\Delta t'} \approx 0 \,, \quad \int_0^t dt'\, e^{\pm 2i\Delta t'} \approx 0$$

$$\hat U_D(t) \underset{\uparrow}{\approx} 1 - it\frac{|g|^2}{\Delta}\Big(\underbrace{\hat a\,\hat a^\dagger}_{\hat a^\dagger \hat a + 1}\sigma_+\sigma_- - \hat a^\dagger \hat a\, \sigma_-\sigma_+\Big) + \dots \overset{!}{\underset{\uparrow}{=}} e^{-\frac{i}{\hbar}\hat H_{eff}\, t}$$

time independent approximation

reinterpretation of perturbative 2nd order result

$$\hat H_{eff} = \hbar\frac{|g|^2}{\Delta}\Big(\underbrace{\sigma_+\sigma_-} + \underbrace{\hat a^\dagger \hat a}\,\sigma_3\Big)$$

energy shift in absence of any photon

depends on photon number

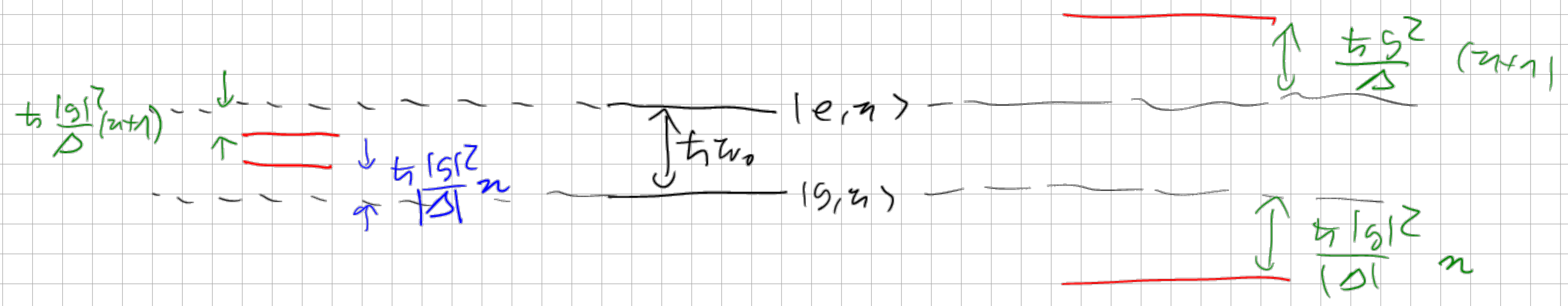
≙ cavity-induced atomic Kerr effect

$$\hat H_{eff}\,|e,n\rangle = +\hbar\,\chi\,(1+n)\,|e,n\rangle \quad ; \quad \chi = \frac{|g|^2}{\Delta}$$

$$\hat H_{eff}\,\underbrace{|g,n\rangle} = -\hbar\,\chi\, n\,|g,n\rangle$$

bare states ≙ eigenstates

$\Rightarrow$ no transitions between $|e\rangle$, $|g\rangle$

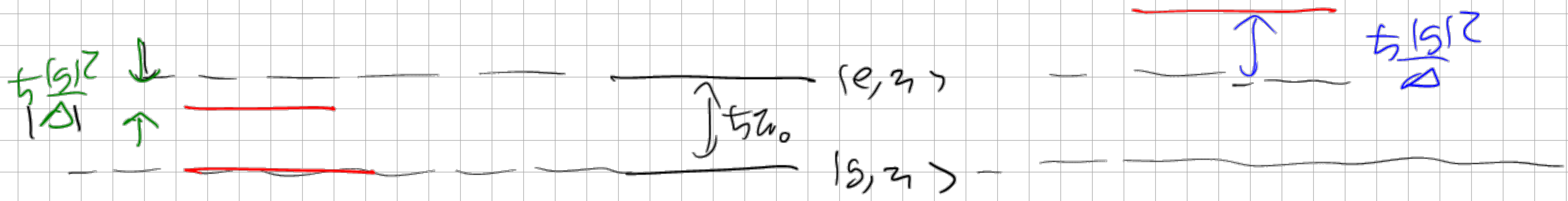



$g \neq 0,\ \Delta < 0$ — presence of photons

$g = 0$

$g \neq 0,\ \Delta > 0$

absence of photons




$g\neq 0,\ \Delta<0$ $\qquad$ $g=0$ $\qquad$ $g\neq,\ \Delta>0$

Consequences of $\hat{H}_{eff}$ for dynamics of particular states:

1) Fock states: trivial

$$e^{-\frac{i}{\hbar}\hat{H}_{eff}t}|g,n\rangle = e^{+i\chi n t}|g,n\rangle$$

$$e^{-\frac{i}{\hbar}\hat{H}_{eff}t}|e,n\rangle = e^{-i\chi(n+1)t}|e,n\rangle$$

only phase factors appear

2) Coherent states: non-trivial, $\quad |\alpha\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}e^{-\frac{|\alpha|^2}{2}}|n\rangle$

$$e^{-\frac{i}{\hbar}\hat{H}_{eff}t}|g,\alpha\rangle = |g\rangle\,\underbrace{e^{i\chi\hat{a}^{\dagger}\hat{a}t}|\alpha\rangle}$$

$$=\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}e^{i\chi n t}\,e^{-\frac{|\alpha|^2}{2}}|n\rangle$$

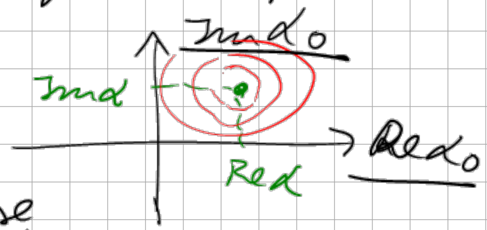
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\big(\underbrace{\alpha e^{i\chi t}}_{=\alpha(t)}\big)^n e^{-\frac{1}{2}|\underbrace{\alpha e^{i\chi t}}_{=\alpha(t)}|^2}|n\rangle$$

$$= |g\rangle |\alpha e^{i\chi t}\rangle = |g, \alpha e^{i\chi t}\rangle$$

$$e^{-\frac{i}{\hbar}\hat{H}_{\text{eff}} t} |e,\alpha\rangle = \dots = e^{-i\chi t} |e, \alpha e^{-i\chi t}\rangle$$

Illustration of dynamics in phase-space: Husimi function

$$Q_{|\alpha\rangle}(\alpha_0) = \frac{1}{\pi} e^{-|\alpha-\alpha_0|^2}$$




$|g,\alpha\rangle \hat{=} \alpha \to \alpha e^{i\chi t} \hat{=}$ anti-clockwise rotation ($\chi > 0$)

$|e,\alpha\rangle \hat{=} \alpha \to \alpha e^{-i\chi t} \hat{=}$ clockwise rotation ($\chi > 0$)

Initial condition: $|\psi(0)\rangle = \underbrace{|\psi_A\rangle}\, |\alpha\rangle$

$$= c_g |g\rangle + c_e |e\rangle$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_{\text{eff}} t} |\psi(0)\rangle = c_g |g, \alpha e^{i\chi t}\rangle + c_e e^{-i\chi t} |e, \alpha e^{-i\chi t}\rangle \Leftarrow$$

$\Rightarrow$ entanglement between light and matter

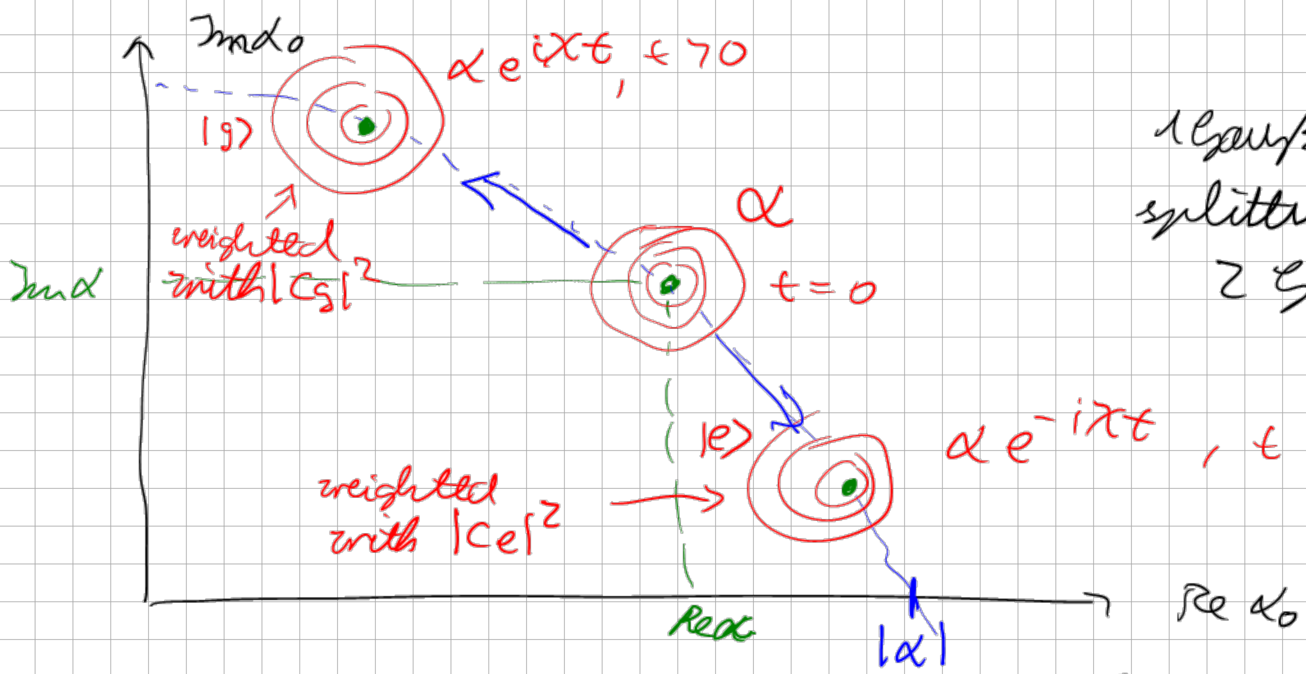
Photonic reduced density matrix:

$$\hat{\rho}_F(t) = \mathrm{Tr}_A\left[\hat{\rho}(t)\right] = \langle g| \underbrace{\hat{\rho}(t)}_{|\psi(t)\rangle\langle\psi(t)|} |g\rangle + \langle e| \underbrace{\hat{\rho}(t)} |e\rangle$$

$$= |c_g|^2 |g, \alpha e^{i\chi t}\rangle\langle g, \alpha e^{i\chi t}| + |c_e|^2 |e, \alpha e^{-i\chi t}\rangle\langle e, \alpha e^{-i\chi t}|$$

$$Q_{\hat{\rho}_F(t)}(\alpha_0) = \frac{1}{\pi}\left\{ |c_g|^2 e^{-|\alpha_0 - \alpha e^{i\chi t}|^2} + |c_e|^2 e^{-|\alpha_0 - \alpha e^{-i\chi t}|^2} \right\}$$

$$Q_{\hat{\varsigma}_F(0)}(\alpha_0) = \frac{1}{\pi} e^{-|\alpha-\alpha_0|^2} \left( |c_g|^2 + |c_e|^2 = 1 \right)$$
$\uparrow t = 0$




1 Gauß function splitting up into 2 Gauß functions

Extreme situation: 2 Gauß functions far away from each other: $t_0 \chi = \frac{\pi}{2}$

$$e^{\pm i\chi t_0} = e^{\pm i\frac{\pi}{2}} = \pm i$$

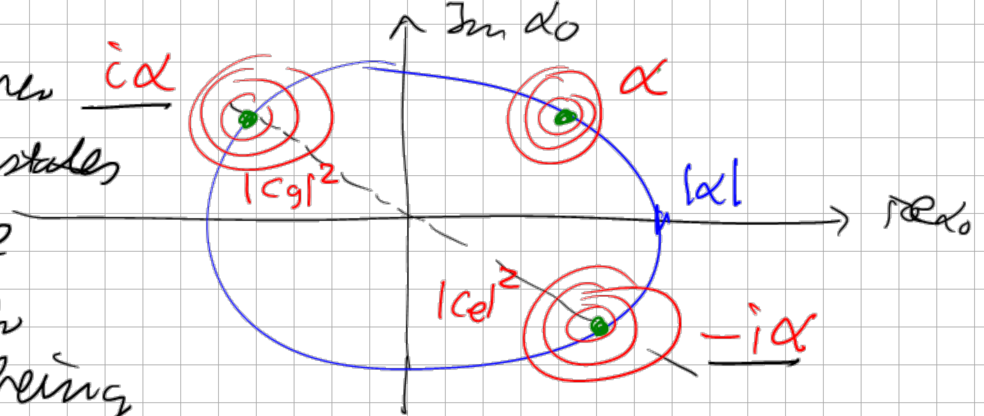
$$Q_{\hat{\varsigma}_F(t_0)}(\alpha_0) = \frac{1}{\pi}\left[ |c_g|^2 e^{-|\alpha_0 - i\alpha|^2} + |c_e|^2 e^{-|\alpha_0 + i\alpha|^2} \right]$$

$|\alpha| \gg 1$: large initial photon number

$\rightarrow$ no overlap between two coherent states

$\rightarrow$ macroscopically distinguishable

Entangled state of light and matter corresponds to a Schrödinger cat being



in entanglement between alive and dead and a decayed and non-decayed radioactive atom

$|\Psi(t)\rangle = c_g\,|\text{atom decayed}\rangle|\text{cat alive}\rangle + c_e\,|\text{atom non-decayed})$

## 4.6 Cavity QED:

$\hat{=}$ interaction of two-level system with quantized electromagnetic field in cavity

experimental realization of $\mathcal{H}$ model is not in optical but in microwave regime

$\Rightarrow$ Rydberg atom in a microwave cavity

photon should live long enough in cavity: quantified by Q-factor

$\omega$: resonator frequency

$\Delta\omega$: bandwidth

$$\left.\begin{array}{l}\omega \\ \Delta\omega\end{array}\right\} \quad Q = \frac{\omega}{\Delta\omega}$$

high Q factor $\hat{=}$ low damping $\hat{=}$ long photon lifetime

### 4.6.1 Rydberg atom:

Rydberg atom = alkali metal atom, where electron is in an excited state with a large principal quantum number $n$ (ca. $n=50$)

binding energy $\quad E_{n\ell} = -\dfrac{Ry}{(n-\delta_\ell)^2}, \quad Ry = 13.6\,eV$

quantum defect: measures deviation from hydrogen atom

$\ell$ small: $\delta_\ell$ is order unity

$\ell$ large: $\delta_\ell$ is negligible $\rightarrow$ binding energy close to hydrogen atom

simulate hydrogen atom for large $\ell$, thus large $n$ is necessary e.g. with Rubidium atoms (87 electrons)

highest possible value of $l$ is $l = n-1$, $m = -(n-1), \dots, +(n-1)$

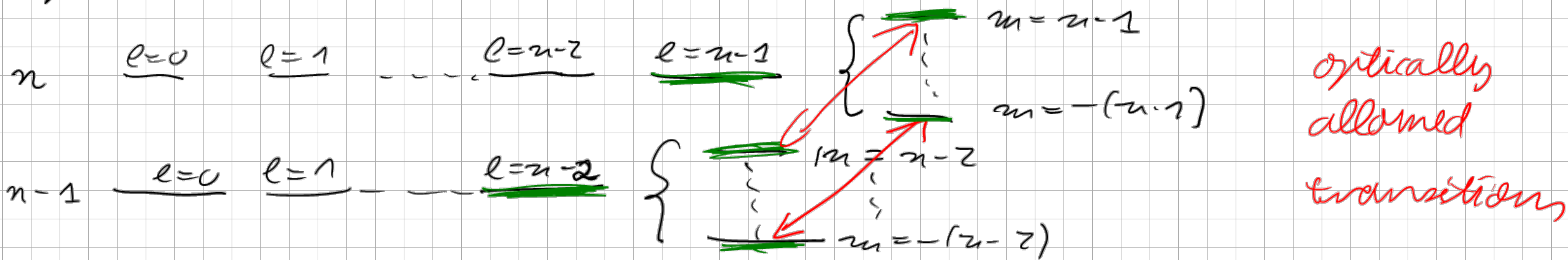
circular Rydberg states : $l = n-1$, $m = \pm(n-1)$

$\hat{=}$ classical limit of electron in circular orbit

Cavity QED relies on circular Rydberg states to due various properties:

(1) Close approximation of 2-level system

dipole moment selection rules: $\Delta l = \pm 1$, $m = 0, \pm 1$




dipole transition of one circular orbit involves a circular orbit of neighbouring principal quantum number

(2) dipole moment $d = \langle n|\hat{d}|n-1\rangle \sim e\, a_n$, $a_n = \boxed{n^2}\, \underline{a_B}$

Bohr radius

(3) spontaneous emission rate

$$\Gamma = \frac{d^2 \omega_0^3}{3\pi \varepsilon_0 \hbar c^3} \sim d^2 \omega_0^3$$

$$\omega_0 = \frac{E_{nl} - E_{n'l'}}{\hbar} \overset{\delta_l \approx 0}{=} \frac{R_y}{\hbar}\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right] \overset{n \gg 1}{\sim} \frac{1}{n^3}$$

$$\Gamma \sim (n^2)^2 (n^{-3})^3 \Gamma_0 = n^{-5}\Gamma_0, \quad \Gamma_0 = 10^9 \frac{1}{s}, \quad \tau = \frac{1}{\Gamma}, \quad \tau_0 = \frac{1}{\Gamma_0} = 1ns \Rightarrow \boxed{\overset{n=50}{\tau} = 0.1 s}$$