Chapter 4: Emission and Absorption of Light by Matter

Overview:

Sect. 4.1 general aspects of light–matter interaction
→ dipole approximation, radiation gauge
→ electromagnetic field interacts only via the electric field via a coupling to the electric dipole moment matter

Note: matter ≙ quantum mechanical; light ≙ classical, quantum mechanical

	light classical	light quantum mechanical
perturbative treatment: Fermi's golden rule	Sect. 4.2 transition amplitudes for emission and absorption are equal	Sect. 4.3 3 elementary processes of Einstein: induced and spontaneous emission, absorption; spont. emission also in vacuum ⇒ quantum mechanical
nonperturbative treatment: restriction to two levels	Sect. 4.4 Rabi model • equivalent to spin 1/2 interacting with magnetic field • optical Bloch equations	Sect. 4.5 Jaynes–Cummings model • more complicated dynamics • resonant light: vacuum Rabi oscillations + collapse and revival • non-resonant light: dressed states, effective dispersive interaction between atoms in a cavity → Sect. 4.6 cavity QED

4.1 Light-Matter Interaction

single electron bound to nucleus via $V(r)$, $r = |\vec{x}|$

$$\hat{H}^{(0)}_{(\vec{x})} = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}\right)^2 + V(r)$$

$$\hat{H}^{(0)}\,\psi^{(0)}_n(\vec{x}) = E^{(0)}_n\,\psi^{(0)}_n(\vec{x})$$

$$\int d^3x\,\psi^{(0)*}_n(\vec{x})\,\psi^{(0)}_m(\vec{x}) = \delta_{n,m}, \qquad \sum_n \psi^{(0)*}_n(\vec{x})\,\psi^{(0)}_n(\vec{x}') = \delta(\vec{x}-\vec{x}')$$

Presence of $\vec{A}(\vec{x},t)$, $\varphi(\vec{x},t)$: extend $\hat{H}^{(0)}$ via minimal coupling

$$\hat{H}(\vec{x},t) = \frac{1}{2M}\left[\frac{\hbar}{i}\vec{\nabla} - q\vec{A}(\vec{x},t)\right]^2 + V(r) + q\,\varphi(\vec{x},t)$$

see QFT notes, chapter 11

$q = -e$, $e > 0$

$$\begin{cases} \vec{B}(\vec{x},t) = \operatorname{rot}\vec{A}(\vec{x},t) \\ \vec{E}(\vec{x},t) = -\dfrac{\partial\vec{A}(\vec{x},t)}{\partial t} - \vec{\nabla}\varphi(\vec{x},t) \end{cases}$$

gauge transformation:

exercise on problem sheet 10

$$\left.\begin{aligned} \varphi'(\vec{x},t) &= \varphi(\vec{x},t) + \frac{\partial\Lambda(\vec{x},t)}{\partial t} \\ \vec{A}'(\vec{x},t) &= \vec{A}(\vec{x},t) - \operatorname{grad}\Lambda(\vec{x},t) \end{aligned}\right\}$$

time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} = \hat{H}(\vec{x},t)\,\psi(\vec{x},t)$$

Aim: simplify light-matter interaction

Tool: unitary transformation $\quad \hat{U}(\vec{x},t)\,\hat{U}^+(\vec{x},t) = \hat{U}^+(\vec{x},t)\,\hat{U}(\vec{x},t) = 1$

$$\psi'(\vec{x},t) = \hat{U}(\vec{x},t)\,\psi(\vec{x},t) \iff \psi(\vec{x},t) = \hat{U}^+(\vec{x},t)\,\psi'(\vec{x},t)$$

$$i\hbar\frac{\partial}{\partial t}\psi'(\vec{x},t) = \hat{U}(\vec{x},t)\,\hat{H}(\vec{x},t)\,\underbrace{\psi(\vec{x},t)}_{\hat{U}^+(\vec{x},t)\,\psi'(\vec{x},t)} + i\hbar\frac{\partial \hat{U}(\vec{x},t)}{\partial t}\,\underbrace{\psi(\vec{x},t)}_{\hat{U}^+(\vec{x},t)\,\psi'(\vec{x},t)}$$

$$= \hat{H}'(\vec{x},t)\,\psi'(\vec{x},t)$$

$$\boxed{\hat{H}'(\vec{x},t) = \hat{U}(\vec{x},t)\,\hat{H}(\vec{x},t)\,\hat{U}^+(\vec{x},t) + i\hbar\frac{\partial \hat{U}(\vec{x},t)}{\partial t}\,\hat{U}^+(\vec{x},t)}$$

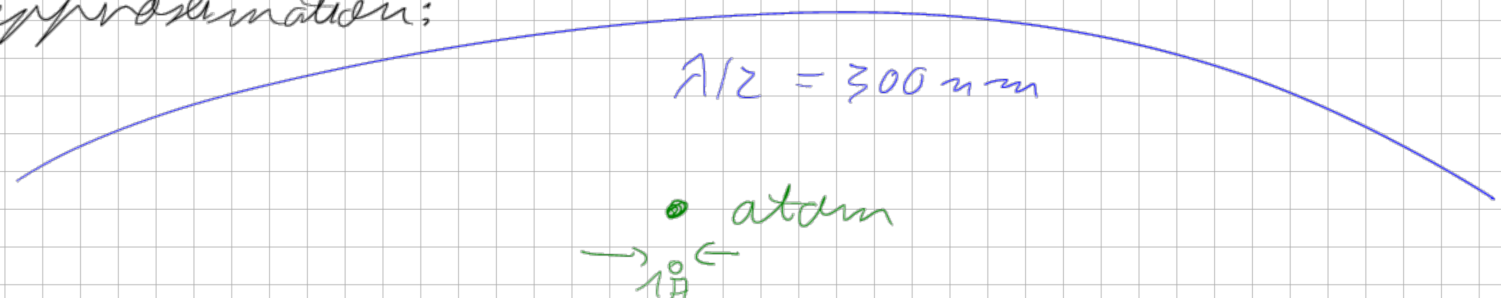
special choice: $\quad \hat{U}(\vec{x},t) = e^{+\frac{i}{\hbar}e\Lambda(\vec{x},t)}, \quad \hat{U}^+(\vec{x},t) = e^{-\frac{i}{\hbar}e\Lambda(\vec{x},t)}$

$$\hat{H}'(\vec{x},t) = \frac{1}{2m}\Big[\frac{\hbar}{i}\vec{\nabla} + \underbrace{e\vec{A}(\vec{x},t) - e\vec{\nabla}\Lambda(\vec{x},t)}_{=e\vec{A}'(\vec{x},t)}\Big]^2 + V(r) \underbrace{- e\varphi(\vec{x},t) - e\frac{\partial\Lambda(\vec{x},t)}{\partial t}}_{=-e\varphi'(\vec{x},t)}$$

$\Rightarrow$ same form as $\hat{H}$ but with $\vec{A}'$ and φ'

Simplify:

- radiation gauge: $\varphi = 0, \quad \operatorname{div}\vec{A} = \vec{0}$
- dipole approximation:



$\rightarrow$ neglect spatial dependence of vector potential $\vec{A}(\vec{x},t) \approx \vec{A}(t)$

- special gauge transformation: $\Lambda(\vec{x},t) := \vec{A}(t)\cdot\vec{x}$

$$\vec{\nabla}\Lambda(\vec{x},t) = \vec{A}(t), \quad \vec{A}'(\vec{x},t) = \vec{A}(t) - \vec{A}(t) \equiv \vec{0}$$

$$\frac{\partial\Lambda(\vec{x},t)}{\partial t} = \frac{\partial\vec{A}(t)}{\partial t}\cdot\vec{x} = -\vec{E}(t)\cdot\vec{x}$$

Transformed Hamiltonian:

$$\hat{H}'(\vec{x},t) = -\frac{\hbar^2}{2M}\Delta + V(r) \boxed{\underbrace{+\, e\,\vec{x}\cdot\vec{E}(t)}_{= -\vec{d}\cdot\vec{E}(t)},} \quad \vec{d} = -e\vec{x}$$

light-matter interaction

Note: This Hamiltonian can be used both for classical and for quantum mechanical light

4.2 Interaction of Atom with Classical Field

$$\vec{E}(t) = \vec{E}_0\cos(\omega t), \quad t \geq 0$$

$t=0$: initial state $\psi_i^{(0)}(\vec{x})$ solving $\hat{H}^{(0)}(\vec{x})\,\psi_i^{(0)}(\vec{x}) = E_i^{(0)}\,\psi_i^{(0)}(\vec{x})$

$t>0$: expand $\psi(\vec{x},t)$ in $\psi_n^{(0)}(\vec{x})$

$$(1) \quad \psi(\vec{x},t) = \sum_n c_n(t)\, e^{-\frac{i}{\hbar}E_n^{(0)}t}\,\psi_n^{(0)}(\vec{x}), \quad \sum_n |c_n(t)|^2 = 1$$

$$(2) \quad i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \Big[\hat{H}^{(0)}(\vec{x}) + \underbrace{\hat{H}^{(P)}(t)}_{= -\vec{d}\cdot\vec{E}(t)}\Big]\,\psi(\vec{x},t)$$

(1) in (2):

$$\sum_n \left\{ i\hbar \frac{\partial c_n(t)}{\partial t} + \cancel{E_n^{(0)} c_n(t)} \right\} e^{-\frac{i}{\hbar}E_n^{(0)}t}\, \psi_n^{(0)}(\vec{x})$$

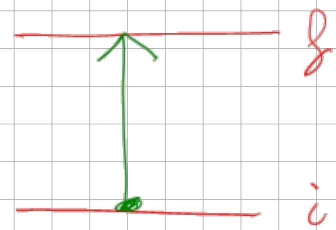
$$= \sum_n \left\{ \cancel{E_n^{(0)}} + \hat{H}^{(p)}(t) \right\} c_n(t)\, e^{-\frac{i}{\hbar}E_n^{(0)}t}\, \psi_n^{(0)}(\vec{x}) \qquad \Big| \int d^3x\, \psi_m^{(0)*}(\vec{x}) \cdot e^{+\frac{i}{\hbar}E_m^{(0)}t}$$

$+ \ (n \leftrightarrow m)$ $\qquad = \int d^3x\, \psi_n^{(0)}(\vec{x})\, \hat{H}^{(p)}(t)\, \psi_m^{(0)}(\vec{x})$

$$i\hbar \frac{\partial}{\partial t} c_n(t) = \sum_m H^{(p)}_{nm}\, c_m(t)\, e^{\frac{i}{\hbar}\underbrace{(E_n^{(0)} - E_m^{(0)})}_{=\hbar\omega_{nm}^{(0)}} t}$$ transition frequencies

initial condition: $c_n(0) = \delta_{ni} = \begin{cases} 1; & n = i \\ 0; & n \neq i \end{cases}$

$t > 0$: i less occupied, $n \neq i$ occupation increases



probability ~~for~~ transition from i to f (final)

$$P_{i\to f}(t) = |c_f(t)|^2$$

coupled differential equations are not solvable exactly
but perturbative treatment is possible → different orders in E_0

$$c_n(t) = c_n^{(0)}(t) + c_n^{(1)}(t) + c_n^{(2)}(t) + \dots$$

valid for small amplitudes

$$\frac{\partial c_n^{(0)}(t)}{\partial t} = 0 \Rightarrow c_n^{(0)}(t) = c_n^{(0)}(0) = \delta_{ni}$$

$$\frac{\partial c_n^{(1)}(t)}{\partial t} = -\frac{i}{\hbar}\sum_m e^{i\omega_{nm}^{(0)} t} H^{(p)}_{nm}(t)\, c_m^{(0)}(t)$$

$$= -\frac{i}{\hbar}\, e^{i\omega_{ni}^{(0)} t}\, H^{(p)}_{ni}(t) \qquad = \delta_{mi}$$

$$\Rightarrow c_n^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt'\, e^{i\omega_{ni}^{(0)} t'}\, H^{(p)}_{ni}(t') \quad \Leftarrow$$

initial state: $n = i$

$$c_i^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt'\, \underbrace{e^{i\underbrace{\omega_{ii}^{(0)}}_{=0} t'}}_{=1}\, \underbrace{H^{(p)}_{ii}(t)}_{=+\int d^3x\, \underbrace{|\psi_i^{(0)}(\vec{x})|^2}_{\text{even parity}}\, \underbrace{(-e\vec{x})}_{\text{odd parity}} \;\equiv 0} = 0$$

$\Rightarrow c_i(t) = c_i^{(0)} = 1$ up to first order

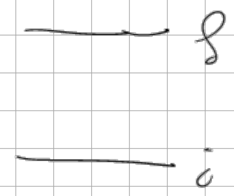
final state: $n = f \neq i$

$$c_f^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt'\, e^{i\omega_{fi}^{(0)} t'}\, \underbrace{H^{(p)}_{fi}(t')}_{= -\vec{d}_{fi}\cdot\vec{E}_0 \cos\omega t'}$$

$$\vec{d}_{fi} = \int d^3x\, \psi_f^{(0)*}(\vec{x})\, \vec{d}\, \psi_i^{(0)}(\vec{x})$$

$$= \frac{i}{2\hbar}\vec{d}_{fi}\vec{E}_0 \left\{ \int_0^t dt'\, e^{i(\omega_{fi}^{(0)}+\omega)t'} + \int_0^t dt'\, e^{i(\omega_{fi}^{(0)}-\omega)t'} \right\}$$

$$= \cancel{\frac{e^{i(\omega_{fi}^{(0)}+\omega)t} - 1}{i(\omega_{fi}^{(0)}+\omega)}} + \frac{e^{i(\omega_{fi}^{(0)}-\omega)t} - 1}{i(\omega_{fi}^{(0)}-\omega)}$$



$\omega > 0,\ \omega_{fi} \geq 0$

~~anti-resonant, small → neglected~~

resonant, large → dominant

⇒ rotating wave approximation

$$c^{(1)}_{fi}(t) = \frac{1}{2\hbar}\, \vec{d}_{fi}\cdot\vec{E}_0\, \frac{e^{i(\omega^{(0)}_{fi}-\omega)t}-1}{\omega^{(0)}_{fi}-\omega}$$

$c^{(0)}_f(0) = 0 \Rightarrow c_f(t) = 0 + c^{(1)}_f(t)$ up to first order

transition probability

$$P_{i\to f}(t) = |c^{(1)}_f(t)|^2 = \frac{|\vec{d}_{fi}\cdot\vec{E}_0|^2}{\hbar^2}\cdot \boxed{\frac{\sin^2(\Delta t/2)}{\Delta^2}}$$

detuning: $\Delta = \omega - \omega^{(0)}_{fi}$

frequency of electromagnetic field

atomic transition frequency between initial and final state

$\Delta > 0$ blue detuned

$\Delta = 0$ resonant

$\Delta < 0$ red detuned