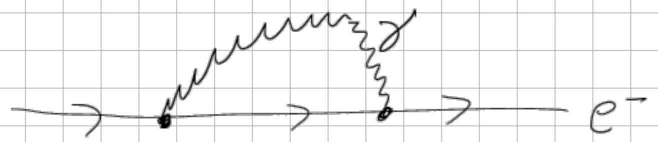
Lamb Shift:

- Phenomenon: s-states get shifted upwards
- Physical origin:



fluctuations due to virtual photons $\hat{=}$ Zitterbewegung

width $\sqrt{\langle \vec{s}^2 \rangle}$

$+$

e^-

- Calculation: $V(\vec{x}) = -\dfrac{e^2}{4\pi\varepsilon_0 |\vec{x}|}$

positive sign

$$\Delta E_L = \langle V(\vec{x}+\vec{s}) \rangle - V(\vec{x}) = \frac{\text{last}}{\text{time}} = \frac{1}{6\pi\varepsilon_0 a_B^3 n^3} \langle \vec{s}^2 \rangle$$

Bohr radius

approximately dependence on principal quantum number can be neglected

$$\frac{\Delta E_L(1^2 S_{1/2})}{\Delta E_L(2^2 S_{1/2})} \approx 8 \iff \frac{8.176\,\text{GHz}}{1.058\,\text{GHz}} \approx 7.7$$

- Still to be determined: $\langle \vec{s}^2 \rangle$

classical approach $\xleftrightarrow[\text{to agree}]{\text{turns out}}$ quantum mechanical approach

2.18.5 Classical Approach:

- Newton equation for displacement $\vec{s}$:

$$m \ddot{\vec{s}}_\omega(t) = -e \vec{E}_\omega(t), \quad \vec{E}_\omega(t) = \vec{E}_0(\omega) e^{-i\omega t}$$

$$\Rightarrow \vec{s}_\omega(t) = \vec{s}_0(\omega) e^{-i\omega t}, \quad \vec{s}_0(\omega) = \frac{e}{m\omega^2} \vec{E}_0(\omega)$$

- Average over all fluctuations: time average

$$\langle \cdot \rangle = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} dt \; \cdot \quad \Rightarrow \langle \vec{s}^2(t) \rangle = \frac{e^2}{m^2 \omega^4} |\vec{E}_0(\omega)|^2 \Leftarrow$$

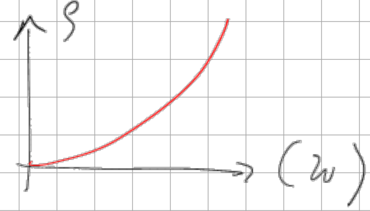
• Both electric and magnetic field contribute equally to zero-point energy

$$\frac{\varepsilon_0}{2}\,|\vec{E}_0(\omega)|^2 = \frac{1}{2}\,\frac{\hbar\omega}{2}\,\underbrace{\varrho(\omega)}\,d\omega$$

number of modes per volume in interval $[\omega, \omega+d\omega]$ per $d\omega$ = mode density

$$N_{\text{modes}} = V\int_{-\infty}^{+\infty} d\omega\, \varrho(\omega) = \underbrace{2}_{\substack{\text{transversal}\\ \text{degrees of freedom}}} V\int \frac{d^3k}{(2\pi)^3} = 2V\int\frac{d^3k}{(2\pi)^3}\,\overbrace{\int_{-\infty}^{+\infty} d\omega\,\delta(\omega - c|\vec{k}|)}^{\equiv 1}$$

$$\Rightarrow \quad \varrho(\omega) = 2\int\frac{d^3k}{(2\pi)^3}\,\delta(\omega - c|\vec{k}|) = 2\,\frac{4\pi}{8\pi^3}\int_0^\infty dk\, k^2\,\delta(\omega - ck) = \frac{1}{\pi^2}\,\frac{\omega^2}{c^3}$$



• Consequence:

$$|\vec{E}_0(\omega)|^2 = \frac{\hbar\omega^3}{2\pi^2\varepsilon_0 c^3}\,d\omega$$

$$\Rightarrow \quad \langle \vec{\xi}^{\,2}\rangle_\omega = \frac{\hbar e^2}{2\pi^2\varepsilon_0 m^2 c^3}\,\frac{d\omega}{\omega}$$

$$\langle \vec{\xi}^{\,2}\rangle = \int_0^\infty d\omega\,\langle\vec{\xi}^{\,2}\rangle_\omega = \frac{\hbar e^2}{2\pi^2\varepsilon_0 m^2 c^3}\int_0^\infty \frac{d\omega}{\omega} \longrightarrow \text{logarithmically divergent}$$

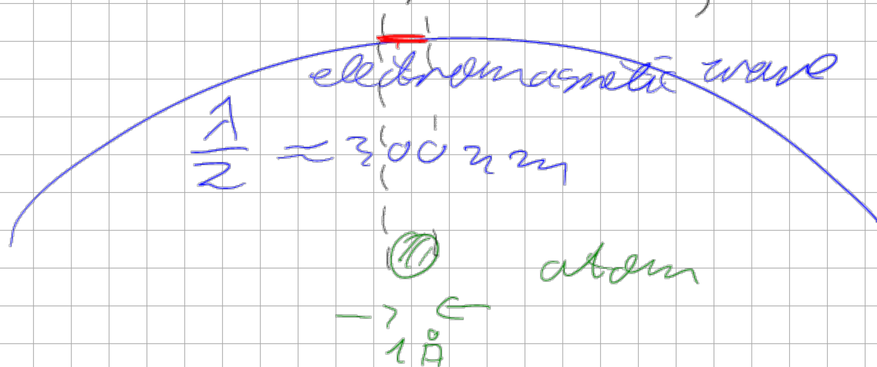
2.18.6 Quantum mechanical Approach:

$$\hat{\vec{E}}(\vec{x},t) = \sum_{\lambda=\pm 1}\int d^3k\,\underbrace{\sqrt{\frac{\hbar\,\omega_{\vec{k}}}{2(2\pi)^3\varepsilon_0}}}_{\text{physical amplitude}}\; i\left\{\vec{\varepsilon}(\vec{k},\lambda)\,e^{i(\boxed{\vec{k}\cdot\vec{x}} - \omega_{\vec{k}}t)}\,a_{\vec{k},\lambda} + h.c.\right\}$$

Wave length occurs in optical range:

$$\lambda = \frac{2\pi}{|\vec{k}|} \approx 400\,\text{nm} - 800\,\text{nm}$$

Extension of atom: $|\vec{x}| = 1\,\text{Å} = 10^{-10}\,\text{m}$



$$|\vec{k}\cdot\vec{x}| \le |\vec{k}|\,|\vec{x}| = \frac{2\pi}{600\,nm}\cdot 10^{-10}\,m \approx 10^{-3} \ll 1$$

Electric field does not change over extension of atom:

→ roughly homogeneous

→ neglect spatial dependence $\Big\}$ called dipole approximation

$$\hat{\vec{E}}(\vec{x},t) = \sum_{\lambda=\pm1}\int d^3k \sqrt{\frac{\hbar\omega_{\vec{k}}}{2(2\pi)^3\varepsilon_0}}\; i\left\{\vec{\varepsilon}(\vec{k},\lambda)\, e^{-i\omega_{\vec{k}}t}\,\hat{a}_{\vec{k},\lambda} + h.c.\right\}$$

"Second quantized" Newton equation:

$$m\,\ddot{\hat{\vec{s}}}(t) = -e\,\hat{\vec{E}}(t)$$

$$\hat{\vec{s}}(t) = \sum_{\lambda=\pm1}\int d^3k\, \frac{e}{m\,\omega_{\vec{k}}^2}\sqrt{\frac{\hbar\omega_{\vec{k}}}{2(2\pi)^3\varepsilon_0}}\; i\left\{\vec{\varepsilon}(\vec{k},\lambda)\, e^{-i\omega_{\vec{k}}t}\,\hat{a}_{\vec{k},\lambda} + h.c.\right\}$$

↑ operator-valued displacement ≙ Zitterbewegung

average ≙ vacuum expectation value:

$$\langle \vec{s}^{\,2}(t)\rangle = \langle 0|\vec{s}^{\,2}(t)|0\rangle = \sum_{\lambda=\pm1}\sum_{\lambda'=\pm1}\int d^3k\int d^3k'\, \frac{-e^2}{m^2\omega_{\vec{k}}^2\omega_{\vec{k}'}^2}\sqrt{\frac{\hbar\omega_{\vec{k}}}{2(2\pi)^3\varepsilon_0}}\sqrt{\frac{\hbar\omega_{\vec{k}'}}{2(2\pi)^3\varepsilon_0}}$$

$$\Big\{\langle 0|\left\{\vec{\varepsilon}(\vec{k},\lambda)e^{-i\omega_{\vec{k}}t}\,\hat{a}_{\vec{k},\lambda} - \cancel{\vec{\varepsilon}^*(\vec{k},\lambda)\, e^{i\omega_{\vec{k}}t}\,\hat{a}^\dagger_{\vec{k},\lambda}}\right\}\left\{\ \ldots\ \cancel{\hat{a}_{\vec{k}',\lambda'}} - \ \ldots\ \hat{a}^\dagger_{\vec{k}',\lambda'}\right\}|0\rangle\Big\}$$

$$\to (-1)\,\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{\varepsilon}^*(\vec{k}',\lambda')\, e^{-i\omega_{\vec{k}}t}\, e^{+i\omega_{\vec{k}'}t}\,\underbrace{\langle 0|\hat{a}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k}',\lambda'}|0\rangle}$$

$$= \underset{\text{correlations}}{\text{like for}} \; \text{-----} = \frac{1}{2\pi^2} \underbrace{\frac{e^2 \hbar}{\varepsilon_0 m^2 c^3}}_{\text{order of magnitude ?}} \underbrace{\int_0^\infty \frac{d\omega}{\omega}}_{\text{How to make sense of this ?}} \longrightarrow \text{same as in classical calculations}$$

$= \delta_{\lambda,\lambda'} \, \delta(\vec{k} - \vec{k}')$

2.18.7 Infrared and Ultraviolet Cut Offs:

Appendix C: Atomic units

$\lambda_c = \frac{2\pi\hbar}{mc}$, $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137} = \frac{\lambda_c}{2\pi a_B}$ Sommerfeld fine structure constant

$$= \frac{\lambda_c^2 \alpha}{\pi}$$

$$\Rightarrow \langle \vec{s}^{\,2} \rangle = \frac{\lambda_c^2 \alpha}{2\pi^3} \int_{\omega_{min}}^{\omega_{max}} \frac{d\omega}{\omega} = \frac{\lambda_c^2 \alpha}{2\pi^3} \underbrace{\ln \frac{\omega_{max}}{\omega_{min}}}_{\text{depends on physical choice of cut-offs}}$$

- Choice of UV cut-off: What is the largest energy scale for this theory?

  $\omega_{max} = \frac{mc^2}{\hbar}$ does not depend on n

- Choice of IR cut-off is less clear, turns out to depend on n

  > first possibilities: frequency of electron on Bohr radius

$$\underbrace{\frac{e^2}{4\pi\varepsilon_0 a_n^2}}_{\text{Coulomb force}} = \underbrace{m\, \omega_{min,n}^{(1)\,2}\, a_n}_{\text{centrifugal force}} \Rightarrow \omega_{min,n}^{(1)} = \ldots = \frac{\alpha c}{a_B} \frac{1}{n^3}$$

a_n → Bohr radius for n

→ second possibilities: frequency of photon on Bohr radius

$$\omega^{(2)}_{min,n} = \frac{2\pi c}{a_n} = \frac{2\pi c}{a_B n^2}$$

- Ratios:

$$\frac{\omega_{max}}{\omega^{(1)}_{min,n}} = \ldots = \frac{n^3}{\alpha^2} \quad , \quad \frac{\omega_{max}}{\omega^{(2)}_{min,n}} = \ldots = \frac{n^2}{2\alpha}$$

- Final result:

$$\Delta E_{L,n} = \frac{8}{3\pi n^3} Ry\, \alpha^3 \ln \frac{\omega_{max}}{\omega_{min,n}}$$

$n=1$: $\Delta E_L(1^2S_{1/2}) \in [0.39, 1.3]$ GHz

|| measured

$\approx$ 0.578 GHz

$n=2$: $\Delta E_L(2^2S_{1/2}) = \frac{1}{16} Ry\, \alpha^2 \cdot F_2$

$$F_2 = \frac{16\alpha}{3\pi} \ln \frac{\omega_{max}}{\omega_{min,n}} \implies F_2 \in [0.05, 0.15]$$

$F_{2,measured} = 0.1$

Document precision for Lamb shift:

experiment (1986): 1 057 845 (9) kHz

theory (1984): 1 057 857 (14) kHz

3 Quantum States of Radiation:

Motivation:

- Vacuum: quantized light can be in different states characterized by $\vec{S}$, 1
- Now: focus discussion on one single mode of electromagnetic field
- Experimental realization: standing wave in cavity at resonance
- Quantum states of single mode field
  - Fock number states $\hat{=}$ stationary states of harmonic oscillator
  - Glauber or coherent state $\hat{=}$ specific states of a harmonic oscillator whose dynamics corresponds to a classical harmonic oscillator
  - Squeezed states:
    > two non-commuting variables
    > One of them has smaller standard deviation, the other one larger standard deviation but Heisenberg uncertainty still valid

  → all pure states
- Example of mixed states: thermal states appear in black-body radiation
- Quantum states can be pictorially represented in phase space which is spanned by two canonical variables
- Due to Heisenberg uncertainty a quantum state can NOT be represented by a point

- no probability for a point in phase space → quasi-probability distribution
- here: Husimi- or Q-function, all states above will be discussed
- Exercise: $\underbrace{\text{P-function}}_{\text{Glauber, Sudarshan}}$, Wigner function W

Announcements:

1) Additional lecture: 25.5., 15.30
2) Two exercises on MO, 05.06. at 9.00 – 10.00, 10.00 – 11.30 (Problem Set 6 + 7)