2 Quantization of Maxwell Field

Motivation:

- All electrodynamic processes are described by Maxwell equations. Surprising the Maxwell theory is a first-quantised theory, although no Planck constant appears
- Resolution of apparent contradiction:
  spatio-temporal derivatives $\leftrightarrow$ Compton wave length

$$\lambda_c = \frac{h}{mc} \quad \rightarrow \quad \frac{1}{\lambda_c} = \frac{mc}{h} \quad \text{appears in wave equation}$$

$\uparrow$ Heisenberg uncertainty

$\frac{mc}{h} \xrightarrow{m \to 0} 0$

$\hbar$ is gone in this limit

s	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	...
$m \neq 0$		$e^{\pm}$, Dirac	$W^{\pm}, Z^0$			
$m = 0$		<u>Weyl</u>	<u>Maxwell</u> photon			

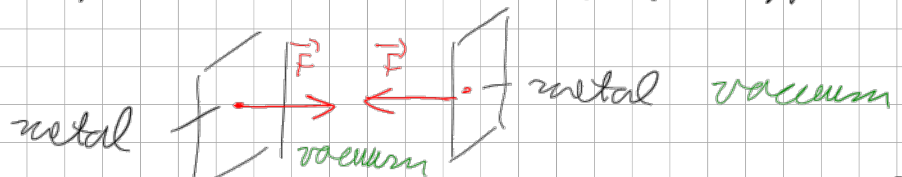
neutrinos were believed to be here

intermediary vector bosons, weak interaction

- Second quantization:
  - Field-theoretical formulation for Maxwell theory: Lagrangian, Hamiltonian
  - Canonical field quantization
  - Problem: local gauge symmetry due to vanishing rest mass of photon
    → choose a particular gauge (Coulomb gauge)

homework

- Three prime examples for quantum fluctuations
  - Vacuum correlation function of electric field: difficult to measure, recently accomplished
  - Casimir effect



  - Lamb shift: degeneracy of 2 1/2 and 2 p 1/2 states in Dirac theory of hydrogen atom is lifted → fine structure in hydrogen atom

reminder

2.1 Maxwell equation:

- Forces of an electromagnetic field on charges at rest or moving are due to electric field strength $\vec{E}$, magnetic induction $\vec{B}$
- Sources for $\vec{E}$, $\vec{B}$: charge density ϱ, current density $\vec{j}$
- Mathematical basis of Maxwell theory: Helmholtz vector decomposition theorem
  Once you know the divergence and the rotation of a vector field, the vector field can be determined by taking into account boundary conditions

$$\vec{V} \iff \operatorname{div} \vec{V}, \ \operatorname{rot} \vec{V}$$

- Electric field strength: Magnetic induction

$\operatorname{div} \vec{E} = \frac{\varrho}{\varepsilon_0}$ (1) ⟵ homogeneous ⟶ $\operatorname{div} \vec{B} = 0$ (2)

$\operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (3) ⟵ inhomogeneous ⟶ $\operatorname{rot} \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$ (4)

$\left.\begin{array}{l}\varepsilon_0 : \text{vacuum dielectric constant}\\ \mu_0 : \text{vacuum permeability}\end{array}\right\}$ light velocity $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

Note: SI units (Système international d'unités)

- Consistency relation: continuity equation:

$$\frac{\partial \varrho}{\partial t} \overset{(1)}{=} \varepsilon_0 \operatorname{div} \frac{\partial \vec{E}}{\partial t} \overset{(4)}{=} \varepsilon_0 \operatorname{div}\left(c^2 \cancel{\operatorname{rot}} \vec{B} - \frac{1}{\varepsilon_0\mu_0}\mu_0 \vec{j}\right) = -\operatorname{div} \vec{j}$$

conservation of charges:

vanishes at infinity

$$\frac{\partial}{\partial t} \underbrace{\int d^3x \, \varrho}_{=Q} = -\int d^3x \operatorname{div} \vec{j} \; \underset{\text{theorem}}{\overset{\text{Gauß}}{=}} \; -\oint d\vec{S} \cdot \vec{j} \; \overset{\downarrow}{=} \; 0$$

2.2 Local gauge symmetry:

- homogeneous Maxwell equations

$\operatorname{div} \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \operatorname{rot} \vec{A}$ ⟵ vector potential

scalar potential

$\operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \operatorname{rot}\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \quad \Rightarrow \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \operatorname{grad} \varphi$

- Inhomogeneous Maxwell equations (1) + (4):

$$\boxed{\begin{array}{l} -\Delta \varphi - \frac{\partial}{\partial t} \operatorname{div} \vec{A} = \frac{\varrho}{\varepsilon_0} \\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \Delta \vec{A} + \operatorname{grad}\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \operatorname{div} \vec{A}\right) = \mu_0 \vec{j} \end{array}}$$

} coupled equations for φ and $\vec{A}$

- Invariance with respect to a local gauge invariance:

$$\boxed{\begin{aligned} \varphi' &= \varphi + \frac{\partial}{\partial t}\Lambda \\ \vec{A}' &= \vec{A} - \operatorname{grad}\Lambda \end{aligned}} \leftarrow \text{arbitrary gauge function}$$

$$\vec{B}' = \operatorname{rot}\vec{A}' = \operatorname{rot}\vec{A} - \cancel{\operatorname{rot}\operatorname{grad}\Lambda} = \operatorname{rot}\vec{A} = \vec{B}$$

$$\vec{E}' = -\operatorname{grad}\varphi' - \frac{\partial \vec{A}'}{\partial t} = -\operatorname{grad}\varphi - \cancel{\operatorname{grad}\frac{\partial}{\partial t}\Lambda} - \frac{\partial\vec{A}}{\partial t} + \cancel{\frac{\partial}{\partial t}\operatorname{grad}\Lambda} = \vec{E}$$

} physics does not change but description of physics does change

- can we choose a gauge such that both coupled equations for φ and $\vec{A}$ decouple
- Coulomb gauge: $\operatorname{div}\vec{A} = 0$ (longitudinal part of $\vec{A}$ vanishes)

$$\Rightarrow \Delta\varphi = -\frac{\rho}{\varepsilon_0} \quad \text{(Poisson eq.)} \quad \Rightarrow (*)\ \varphi(\vec{x}, t) = \int d^3x' \, \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|}$$

instantaneous reaction $\hat{=}$ not manifestly Lorentz invariant → only valid in certain inertial system

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \Delta\vec{A} = \mu_0 \vec{j} - \frac{1}{c^2}\frac{\partial}{\partial t}\operatorname{grad}\varphi$$

Only 2 degrees of freedom left
- Coulomb gauge
- φ is determined

} the remaining 2 degrees of freedom correspond physical to transversal degrees of freedom

- Lorentz gauge: $\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \operatorname{div}\vec{A} = 0$

$$\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\varphi = \frac{\rho}{\varepsilon_0}, \quad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\vec{A} = \mu_0 \vec{j}$$

$\Rightarrow$ Lorentz invariant

Disadvantage: $4 - 1 = 3$ degrees of freedom

but one should only have 2 physical degrees of freedom $\Rightarrow$ to be explained on Monday

- From now on: vacuum $\rho(\vec{x},t) = 0$, $\vec{j}(\vec{x},t) = 0$
- Coulomb gauge: $\operatorname{div} \vec{A} = 0$ (standard in quantum optics) (I) } radiation

(*) $\Rightarrow$ $\varphi(\vec{x},t) \equiv 0$ (II) } gauge

$$\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\vec{A} = 0$$ homogeneous wave equation (Strahlungseichung)

but also $\operatorname{div} \vec{A} = 0$ has to be taken into account

- At the end: $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \operatorname{rot} \vec{A}$

2.3 Lagrange Formulation:

- Aim: variational principle so that Euler-Lagrange equations yield Maxwell equations
- Action: functional of vector potential $\vec{A}(\vec{x},t)$

$$S[\vec{A}(\cdot,\cdot)] = \int dt \underbrace{\int d^3x \; \mathcal{L}}_{\text{Lagrange function}} \leftarrow \text{Lagrange density}$$

Ansatz:

$$\mathcal{L} = \mathcal{L}\left(\underbrace{A_k(\vec{x},t)}_{k = 1,2,3}, \frac{\partial A_k(\vec{x},t)}{\partial t}, \partial_j A_k(\vec{x},t)\right)$$

Hamilton principle:

functional derivative $\frac{\delta S}{\delta A_k(\vec{x},t)} = 0$

— Crash course on functional derivatives:

$$\frac{dx}{dx} = 1 \; , \quad \frac{\partial x_i}{\partial x_j} = \delta_{ij} \; , \quad \frac{\delta x(t)}{\delta x(t')} = \delta(t-t')$$

total derivative, partial derivative, functional derivative

$$\Rightarrow \quad \frac{\delta A_\ell(\vec{x},t)}{\delta A_k(\vec{x}',t')} = \delta_{k\ell}\, \delta(\vec{x}-\vec{x}')\, \delta(t-t')$$

- Greiner, Reinhardt: Field Quantization
- My QFT lecture notes

Einstein summation convention

$$\frac{\delta S}{\delta A_k(\vec{x},t)} = \int dt' \int d^3x' \left\{ \frac{\partial \mathcal{L}}{\partial A_\ell(\vec{x}',t')} \frac{\delta A_\ell(\vec{x}',t')}{\delta A_k(\vec{x},t)} + \frac{\partial \mathcal{L}}{\partial \frac{\partial A_\ell(\vec{x}',t')}{\partial t'}} \underbrace{\frac{\delta \frac{\partial A_\ell(\vec{x}',t')}{\partial t'}}{\delta A_k(\vec{x},t)}}_{= \frac{\partial}{\partial t'} \frac{\delta A_\ell(\vec{x}',t')}{\delta A_k(\vec{x},t)}} + \frac{\partial \mathcal{L}}{\partial[\partial'_j A_\ell(\vec{x}',t')]} \underbrace{\frac{\delta \partial'_j A_\ell(\vec{x}',t')}{\delta A_k(\vec{x},t)}}_{= \partial'_j \frac{\delta A_\ell(\vec{x}',t')}{\delta A_k(\vec{x},t)}} \right\}$$

partial integration

$$= \int dt' \int d^3x' \left\{ \frac{\partial \mathcal{L}}{\partial A_\ell(\vec{x}',t')} - \frac{\partial}{\partial t'} \frac{\partial \mathcal{L}}{\partial \frac{\partial A_\ell(\vec{x}',t')}{\partial t'}} - \partial'_j \frac{\partial \mathcal{L}}{\partial \partial'_j A_\ell(\vec{x}',t')} \right\} \underbrace{\frac{\delta A_\ell(\vec{x}',t')}{\delta A_k(\vec{x},t)}}_{= \delta_{k\ell}\, \delta(\vec{x}-\vec{x}')\, \delta(t-t')} = 0$$

Euler – Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial A_k(\vec{x},t)} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \frac{\partial A_k(\vec{x},t)}{\partial t}} - \partial_j \frac{\partial \mathcal{L}}{\partial \partial_j A_k(\vec{x},t)} = 0$$