Answer of Question from last time

- given $\rho, \vec{j}$ $\Rightarrow$ coupled (partial) differential equations (for) $\varphi, \vec{A}$
- decoupling via choosing Lorentz gauge: $\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \operatorname{div} \vec{A} = 0$

$$\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\varphi = \frac{\rho}{\varepsilon_0}, \quad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\vec{A} = \mu_0 \vec{j}$$

- concrete realization: assume $\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \operatorname{div}\vec{A} \neq 0$

$$\varphi' = \varphi + \frac{\partial \Lambda}{\partial t}, \quad \vec{A}' = \vec{A} - \operatorname{grad}\Lambda$$

$$\frac{1}{c^2}\frac{\partial \varphi'}{\partial t} + \operatorname{div}\vec{A}' = \frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda + \operatorname{div}\vec{A} - \underbrace{\operatorname{div}\operatorname{grad}}_{=\Delta}\Lambda \overset{!}{=} 0$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\Lambda = -\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \operatorname{div}\vec{A}\right) = f \neq 0$$

$\Rightarrow$ particular solution: $\Lambda_{part}(\vec{x}, t) = \frac{1}{4\pi}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} f\left(\vec{x}', \underbrace{t - \frac{|\vec{x}-\vec{x}'|}{c}}_{\text{retardation}}\right)$

- A second degree of freedom turns out to redundant only in case of vacuum: $\rho = 0, \vec{j} = \vec{0}$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\varphi = 0, \quad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\vec{A} = \vec{0}, \quad \frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \operatorname{div}\vec{A} = 0$$

Choose now Λ as a homogeneous solution of wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\Lambda_{hom} = 0, \quad \varphi' = \varphi + \frac{\partial \Lambda_{hom}}{\partial t}, \quad \vec{A}' = \vec{A} - \operatorname{grad}\Lambda_{hom}$$

and demand in addition $\varphi' = 0$ and $\cancel{\frac{1}{c^2}\frac{\partial \varphi'}{\partial t}} + \underbrace{\operatorname{div} \vec{A}' = 0}$

$\rightarrow$ radiation gauge   Coulomb gauge

Back to the Lecture:

Lagrange formalism for Maxwell theory

$$S = S[\vec{A}(\cdot,\cdot)] = \int dt \int d^3x \; \mathcal{L}$$

action

Hamilton principle: $\underbrace{\frac{\delta}{\delta A_k(\vec{x},t)}} S = 0$

functional derivative

Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial A_k(\vec{x},t)} - \partial_j \frac{\partial \mathcal{L}}{\partial[\partial_j A_k(\vec{x},t)]} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \frac{\partial A_k(\vec{x},t)}{\partial t}} = 0$$

Lagrange density of Maxwell theory:

$$\mathcal{L} = \underbrace{\frac{\varepsilon_0}{2}\vec{E}^2}_{\text{electric energy density}} - \underbrace{\frac{1}{2\mu_0}\vec{B}^2}_{\text{magnetic energy density}} = \underbrace{\frac{\varepsilon_0}{2}\left(\frac{\partial \vec{A}}{\partial t}\right)^2}_{\text{"kinetic energy" density}} - \underbrace{\frac{1}{2\mu_0}(\operatorname{rot}\vec{A})^2}_{\text{"potential energy" density}}$$

$$(\operatorname{rot}\vec{A})^2 = \varepsilon_{jkl}\,\partial_k A_l\, \varepsilon_{jmn}\,\partial_m A_n$$

$$= \underbrace{\varepsilon_{klj}\,\varepsilon_{mnj}}_{= \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}} \partial_k A_l\, \partial_m A_n = \boxed{\partial_k A_l\, \partial_k A_l} - \underbrace{(\partial_k A_l)\,\partial_l A_k}$$

$$= \underbrace{\partial_k (A_\ell \partial_\ell A_k)}_{=0 \text{ Gauß law}} - A_\ell \underbrace{\partial_k \partial_\ell A_k}_{= \partial_\ell \underbrace{\partial_k A_k}_{=0 \text{ Coulombgauge}}}$$

$$\Rightarrow \quad \mathcal{L} = \frac{\varepsilon_0}{2} \frac{\partial A_\ell}{\partial t} \frac{\partial A_\ell}{\partial t} - \frac{1}{2\mu_0} \partial_k A_\ell \, \partial_k A_\ell \qquad \text{(vacuum)}$$

$$\frac{\partial \mathcal{L}}{\partial A_k(\vec{x},t)} = 0 \, , \quad \frac{\partial \mathcal{L}}{\partial \frac{\partial A_k(\vec{x},t)}{\partial t}} = \varepsilon_0 \frac{\partial A_k(\vec{x},t)}{\partial t} \, , \quad \frac{\partial \mathcal{L}}{\partial \, \partial_j A_k(\vec{x},t)} = -\frac{1}{\mu_0} \partial_j A_k(\vec{x},t)$$

$$- \varepsilon_0 \frac{\partial^2 A_k(\vec{x},t)}{\partial t} + \frac{1}{\mu_0} \underbrace{\partial_j \partial_j}_{=\Delta} A_k(\vec{x},t) \Rightarrow \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \vec{A} = \vec{0}$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Note: $\operatorname{div} \vec{A} = 0$ (Coulomb gauge)

2.4 Hamilton Formalism:

$$\Pi_k(\vec{x},t) = \frac{\delta A}{\delta \frac{\partial A_k(\vec{x},t)}{\partial t}} = \frac{\partial \mathcal{L}}{\partial \frac{A_k(\vec{x},t)}{\partial t}} = \underbrace{\varepsilon_0}_{\text{"mass"}} \underbrace{\frac{\partial A_k(\vec{x},t)}{\partial t}}_{\text{"velocity"}} \Rightarrow \frac{\partial A_k}{\partial t} = \frac{1}{\varepsilon_0} \Pi_k$$

"momentum"

$\rightarrow$ "mechanical analogy"

Legendre transformation:

Lagrange density

$$\mathcal{H} = \Pi_k \frac{\partial A_k}{\partial t} - \mathcal{L} = \frac{1}{\varepsilon_0} \Pi_k \Pi_k - \left(\frac{\varepsilon_0}{2} \frac{1}{\varepsilon_0^2} \Pi_k \Pi_k - \frac{1}{2\mu_0} \partial_k A_\ell \, \partial_k A_\ell \right)$$

$$= \underbrace{\frac{1}{2\varepsilon_0} \Pi_k \Pi_k}_{\text{"}\frac{p^2}{2m}\text{"}} + \frac{1}{2\mu_0} \partial_k A_\ell \, \partial_k A_\ell$$

Hamilton density

$$H = \int d^3x\, \mathcal{H} = \int d^3x \left\{ \frac{1}{2\varepsilon_0} \Pi_k \Pi_k + \frac{1}{2\mu_0} \partial_k A_l \partial_k A_l \right\}$$

2.5 Canonical Field Quantization:

$A_j(\vec{x},t),\ \Pi_j(\vec{x},t) \implies \hat{A}_j(\vec{x},t),\ \hat{\Pi}_j(\vec{x},t)$ (Heisenberg picture)

equal-time commutation relations

Pauli spin statistics theorem:

integer spin	half-integer spin
$\Rightarrow$ boson	$\Rightarrow$ fermions
commutation relations	anticommutation relations

$$[\hat{A}_k(\vec{x},t), \hat{A}_l(\vec{x}',t)]_- = 0 = [\hat{\Pi}_k(\vec{x},t), \hat{\Pi}_l(\vec{x}',t)]_- = 0$$

$$[\hat{A}, \hat{B}]_- = \hat{A}\hat{B} - \hat{B}\hat{A}$$

intriguing problem: $[\hat{A}_k(\vec{x},t), \hat{\Pi}_l(\vec{x}',t)]_- = ?$

naive approach: Coulomb gauge $= i\hbar\, \delta_{kl}\, \delta(\vec{x}-\vec{x}')$ $(*)$

problem: $\operatorname{div} \vec{A}(\vec{x},t) = 0 \implies \operatorname{div} \hat{\vec{A}}(\vec{x},t) = 0$ Coulomb gauge

$\partial_k\ (*)$: left $[\underbrace{\partial_k \hat{A}_k(\vec{x},t)}_{=0}, \hat{\Pi}_l(\vec{x}',t)]_-$

right $i\hbar\, \partial_k \delta_{kl}\, \delta(\vec{x}-\vec{x}') = i\hbar\, \partial_l \delta(\vec{x}-\vec{x}') \neq 0$

Constructive solution:

$$[\hat{A}_k(\vec{x},t), \hat{\Pi}_l(\vec{x}',t)]_- = i\hbar \underbrace{\delta^T_{kl}(\vec{x}-\vec{x}')}_{\text{transversal delta function}} \qquad (**)$$

Coulomb gauge — transversal delta function is still to be determined

$\partial_k (**)$: $\quad 0 = \cancel{i\hbar}\, \partial_k \delta^T_{kl}(\vec{x}-\vec{x}')$

solution: Fourier transformation

$$\delta^T_{kl}(\vec{x}-\vec{x}') = \int \frac{d^3k}{(2\pi)^3}\, \delta^T_{kl}(\vec{k})\, e^{i\vec{k}(\vec{x}-\vec{x}')}$$

$$\partial_k \delta^T_{kl}(\vec{x}-\vec{x}') = \int \frac{d^3k}{(2\pi)^3}\, \delta^T_{kl}(\vec{k})\, \underbrace{\partial_k e^{i\vec{k}(\vec{x}-\vec{x}')}}_{= i k_k e^{i\vec{k}(\vec{x}-\vec{x}')}} = 0$$

$$\Rightarrow \quad k_k\, \delta^T_{kl}(\vec{k}) = 0 \qquad \text{transversality condition}$$

solution ansatz: $\quad \delta^T_{kl}(\vec{k}) = \delta_{kl} + k_k k_l\, f(\vec{k})$

$$k_k \delta^T_{kl}(\vec{k}) = \delta_{kl} k_k + k_k k_k k_l f(\vec{k})$$
$$= k_l + \vec{k}^2 k_l f(\vec{k}) = k_l \{1 + \vec{k}^2 f(\vec{k})\} \stackrel{!}{=} 0$$

$$\Rightarrow f(\vec{k}) = -\frac{1}{\vec{k}^2} \qquad \Rightarrow \delta^T_{kl}(\vec{k}) = \delta_{kl} - \frac{k_k k_l}{|\vec{k}|^2}$$

$$\delta^T_{kl}(\vec{x}-\vec{x}') = \int \frac{d^3k}{(2\pi)^3} \left\{ \delta_{kl} - \frac{k_k k_l}{\vec{k}^2} \right\} e^{i\vec{k}(\vec{x}-\vec{x}')}$$

$$= \delta_{kl}\, \underline{\delta(\vec{x}-\vec{x}')} - \int \frac{d^3k}{(2\pi)^3} \frac{1}{\vec{k}^2} (-i\partial_k)(-i\partial_l)\, e^{i\vec{k}(\vec{x}-\vec{x}')}$$

$$\partial_k e^{i\vec{k}(\vec{x}-\vec{x}')} = i k_k\, e^{i(\vec{x}-\vec{x}')}$$

$$+\,\partial_k \partial_l \underbrace{\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} e^{i\vec{k}(\vec{x}-\vec{x}')}}_{=\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|}}$$

Caveat: heuristic derivation, has to be crosscheck with measurements

2.6 Heisenberg equation:

Hamilton operator:

$$\hat{H} = \frac{1}{2}\int d^3x' \left\{ \frac{1}{\varepsilon_0} \hat{\pi}_k(\vec{x}',t)\hat{\pi}_k(\vec{x}',t) + \frac{1}{\mu_0} \partial_k' \hat{A}_l(\vec{x}',t)\, \partial_k' \hat{A}_l(\vec{x}',t) \right\}$$

Note: no operator ordering problem!

$$i\hbar \frac{\partial \hat{A}_j(\vec{x},t)}{\partial t} = [\hat{A}_j(\vec{x},t), \hat{H}]_- = \frac{1}{2\varepsilon_0}\int d^3x' \, [\hat{A}_j(\vec{x},t), \hat{\pi}_k(\vec{x}',t)\hat{\pi}_k(\vec{x}',t)]_-$$

$$= \frac{1}{2\varepsilon_0} 2 \int d^3x' \underbrace{[\hat{A}_j(\vec{x},t), \hat{\pi}_k(\vec{x}',t)]_-}_{= i\hbar\, \delta^T_{jk}(\vec{x}-\vec{x}')} \hat{\pi}_k(\vec{x}',t)$$

$$[\hat{A}, \hat{B}\hat{C}]_- = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A} + \hat{B}\hat{A}\hat{C}$$

$$= [\hat{A}, \hat{B}]_- \hat{C} + \hat{B}[\hat{A}, \hat{C}]_-$$

$$= \frac{1}{\varepsilon_0} i\hbar \int d^3x' \left\{ \delta_{jk}\, \delta(\vec{x}-\vec{x}') + \overbrace{\partial_j \partial_k}^{= \partial_j' \partial_k'} \left(\frac{1}{4\pi |\vec{x}-\vec{x}'|} \right) \right\} \hat{\pi}_k(\vec{x}',t)$$

$$= i\hbar \left\{ \frac{1}{\varepsilon_0} \hat{\pi}_j(\vec{x},t) - \int d^3x' \, \partial'_j \frac{1}{4\pi |\vec{x}-\vec{x}'|} \, \partial'_k \hat{\pi}_k(\vec{x}',t) \right\}$$

partial integration (Gauß theorem)

$$\varepsilon_0 \frac{\partial}{\partial t} \underbrace{\partial'_k \hat{A}_k(\vec{x}',t)}_{=0} \quad \text{(Coulomb gauge)}$$

$$\Rightarrow \frac{\partial \hat{A}_j(\vec{x},t)}{\partial t} = \frac{1}{\varepsilon_0} \hat{\pi}_j(\vec{x},t) \quad \checkmark \quad (1)$$

in the same way

$$\cancel{i\hbar} \frac{\partial}{\partial t} \hat{\pi}_j(\vec{x},t) = [\hat{\pi}_j(\vec{x},t), \hat{H}]_- = \cancel{i\hbar} \frac{1}{\mu_0} \Delta \hat{A}_j(\vec{x},t)$$

$$\Rightarrow \frac{\partial \hat{\pi}_j(\vec{x},t)}{\partial t} = \frac{1}{\mu_0} \Delta \hat{A}_j(\vec{x},t) \quad (2)$$

$$(1)+(2): \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \Rightarrow \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \hat{\vec{A}}(\vec{x},t) = 0$$

homogeneous wave equation