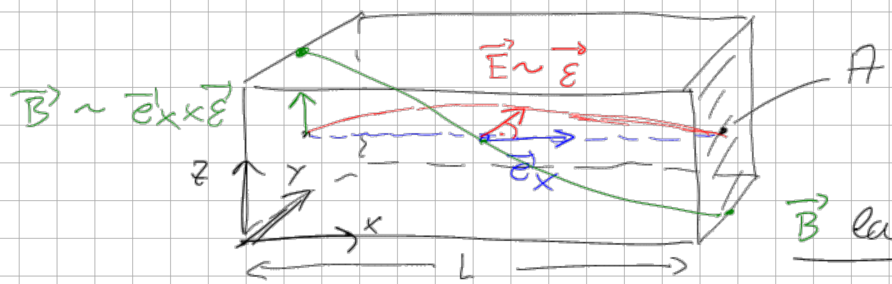
3.1 Single Cavity Mode:



cavity volume

$V = A \cdot L$

modes for

$k_x = \frac{\pi}{L}$

$\vec{B}$ lags behind $\vec{E}$ by a quarter of a period

Vacuum:

$$\hat{\vec{E}}(\vec{x},t) = \sum_{\lambda=\pm1} \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\hbar\omega_k}{2(2\pi)^3\varepsilon_0}}\, i\left\{\vec{\varepsilon}(\vec{k},\lambda)\, e^{i(\vec{k}\vec{x}-\omega_k t)}\, \hat{a}_{\vec{k}\lambda} + h.c.\right\}$$

$\downarrow$ one mode: $\vec{k}, \lambda$

$$\hat{\vec{E}}(\vec{x},t) = \underbrace{\sqrt{\frac{\hbar\omega}{2(2\pi)^3\varepsilon_0}}}_{=E_0}\, i\left\{\vec{\varepsilon}\, e^{i(\vec{k}\vec{x}-\omega t)}\,\hat{a} - \vec{\varepsilon}^{\,*}\, e^{-i(\vec{k}\vec{x}-\omega t)}\,\hat{a}^{+}\right\}$$

Different situation in cavity:

TM mode:

$$\vec{E}(\vec{x},t) = \underbrace{N_{TM}}_{=\frac{E_0}{k_x k_\perp}}\left\{ i k_x \sin(k_x x)\, \vec{k}_\perp - \cancel{\vec{k}_\perp^2 \cos(k_x x)\, \vec{e}_x}\right\} e^{i(\vec{k}_\perp\vec{x}-\omega t)}$$

$$\vec{\varepsilon} = \lim_{k_\perp \to 0} \frac{\vec{k}_\perp}{k_\perp}$$

$\vec{k}_\perp \downarrow 0$ $\longrightarrow$

$$\vec{E}(\vec{x},t) = i E_0 \sin(k_x x)\, \vec{\varepsilon}\, e^{-i\omega t}$$

$e^{-i(\omega t - \frac{\pi}{2})}$

$$\vec{B}(\vec{x},t) = \underbrace{N\,\tau\, m}_{=\frac{E_0}{k_x (k_\perp)}} \frac{\omega}{c^2} \cos(k_x x)\, \vec{e}_x \times \vec{k}_\perp\, e^{i(\vec{k}_\perp \vec{x} - \omega t)}$$

$$\omega = \underbrace{|\vec{k}|}_{k_x} \cdot c \quad (\vec{k}_\perp = \vec{0})$$

$\vec{k}_\perp \to 0$

$$\longrightarrow \frac{E_0}{c}\, \vec{e}_x \times \vec{\varepsilon}\, \cos(k_x x)\, e^{-i\omega t}$$

Second quantization:

$$\hat{\vec{E}}(\vec{x},t) = i E_0\, \vec{\varepsilon}\,\left(e^{-i\omega t}\hat{a} - \hat{a}^+ e^{i\omega t}\right) \sin(k_x x)$$

$$\hat{\vec{B}}(\vec{x},t) = \frac{E_0}{c}\, \vec{e}_x \times \vec{\varepsilon}\,\left(\hat{a}e^{-i\omega t} + \hat{a}^+ e^{i\omega t}\right) \cos(k_x x)$$

E_0 still unknown

Vacuum:

$$E_0 = \sqrt{\frac{\hbar\omega}{2(2\pi)^3 \varepsilon_0}} \longrightarrow$$

Cavity:

$$E_0 = \sqrt{\frac{\hbar\omega}{V \varepsilon_0}}$$

Hamiltonian:

$$\frac{1}{2}\varepsilon_0 \hat{\vec{E}}^2 = \frac{\varepsilon_0}{2} E_0^2 \sin^2(k_x x)\left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\; \boxed{- \hat{a}^2 e^{-2i\omega t} - \hat{a}^{+2} e^{2i\omega t}}\right)$$

$$\frac{1}{2\mu_0}\hat{\vec{B}}^2 = \frac{\varepsilon_0 E_0^2}{2\cancel{\mu_0 c^2}} \cos^2(k_x x)\left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\; \boxed{+ \hat{a}^2 e^{-2i\omega t} + \hat{a}^{+2} e^{2i\omega t}}\right)$$

cancellation of time dependent terms

$$\hat{H} = A \int_0^L dx \left\{ \frac{\varepsilon_0}{2} \hat{\vec{E}}^2(\vec{x},t) + \frac{\hat{\vec{B}}^2(\vec{x},t)}{2\mu_0} \right\}$$

$$\int_0^L dx \sin^2(k_x x) = \frac{L}{2} = \int_0^L dx \cos^2(k_x x)$$

$$\hat{H} = A \frac{L}{\cancel{2}} \frac{\varepsilon_0}{\cancel{2}} E_0^2\, \cancel{2} \left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right) = V \varepsilon_0 \overbrace{E_0^2}^{=\frac{\hbar\omega}{V\varepsilon_0}} \left(\hat{n} + \frac{1}{2}\right) = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$$

$$\hbar \omega \cdot \big(\underbrace{\hat{n}}_{=\hat{a}^+\hat{a}} + \tfrac{1}{2} \big)$$

Remark:

intensity: $I_0 = \frac{\hbar \omega c}{V}$

$A = (1 \mu m)^2$, $L = 1 cm$, $n = 1$ (ground mode) $\Rightarrow I_0 = 0.3 \frac{W}{m^2} \approx 10^{-4} I_E$

solar constant: mean solar electromagnetic radiation arriving at surface of earth

$$I_E = 1.361 \frac{kW}{m^2}$$

3.2 Fock states of Single Mode: see Chapter, Appendix A

quantum states: $|n\rangle$; $n = 0, 1, 2, \ldots$

quantum number = photon number

$\Rightarrow$ Fock states, number states

eigenstates of photon number operator: $\hat{n}|n\rangle = n|n\rangle$

properties of Fock states

1) $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$
   $\hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle$
   $\Rightarrow \langle n'|\hat{a}|n\rangle = \sqrt{n}\, \delta_{n', n-1}$
   $\langle n'|\hat{a}^+|n\rangle = \sqrt{n+1}\, \delta_{n', n+1}$

2) $|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$, vacuum state: $\hat{a}|0\rangle = 0$, $\langle 0|\hat{a}^+ = 0$

3) orthonormality $\langle n|n'\rangle = \delta_{n,n'}$

4) completeness: $\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$

$\Rightarrow$ Hilbert space spanned by $\{|n\rangle\}_{n=0}^{\infty}$

Fock states are non-classical states, difficult to prepare experimentally

student talk: New J. Phys. 6, 97 (2004)

Expectation value: $\langle \cdot \rangle_n = \langle n | \cdot | n \rangle$

$\langle n | \hat{n} | n \rangle = \langle \hat{n} \rangle_n = n$

$\langle \hat{n}^2 \rangle_n = \langle n | \hat{n}^2 | n \rangle = n^2$

variance: $\langle \Delta n^2 \rangle_n = \langle \hat{n}^2 \rangle_n - \langle \hat{n} \rangle_n^2 = 0$

physical interpretation:

- Energy is determined by photon number
- Energy, i.e. photon number, is sharply defined
- No energy (photon number) fluctuations occurring

3.3 Quadratures:

family of operators: $\hat{X}_{\Theta} = \frac{1}{\sqrt{2}}\left(\hat{a} e^{-i\Theta} + \hat{a}^{+} e^{+i\Theta} \right)$

hermiticity: $\hat{X}_{\Theta} = \hat{X}^{+}_{\Theta}$

$\Rightarrow$ appear quite often in quantum optics

1) Example: single cavity mode

$\hat{\vec{E}}(\vec{x},t) = \sqrt{2} E_0 \vec{\varepsilon} \hat{X}_{\omega t - \frac{\pi}{2}} \sin(k_x x) \sim \hat{X}_{\omega t - \frac{\pi}{2}} = \underline{\omega(t - \frac{\pi}{2\omega})}$

$\vec{B}(\vec{x},t) = \sqrt{2} E_0 \vec{e}_x \times \vec{\varepsilon} \hat{X}_{\omega t} \cos(k x) \sim \hat{X}_{\underline{\omega t}}$ $\longrightarrow$ $\vec{B}$ lags behind $\vec{E}$ by a quarter period

2) Example: harmonic oscillator $\hat{H} = \hbar\omega(\hat{a}^{+}\hat{a} + \frac{1}{2})$

$i\hbar \frac{\partial}{\partial t} \hat{a}(t) = [\hat{a}(t), \hat{H}]_{-} = \hbar\omega \hat{a}(t) \Rightarrow \hat{a}(t) = \hat{a}\, e^{-i\omega t}$

same way: $\hat{a}^+(t) = \hat{a}^+ e^{i\omega t}$

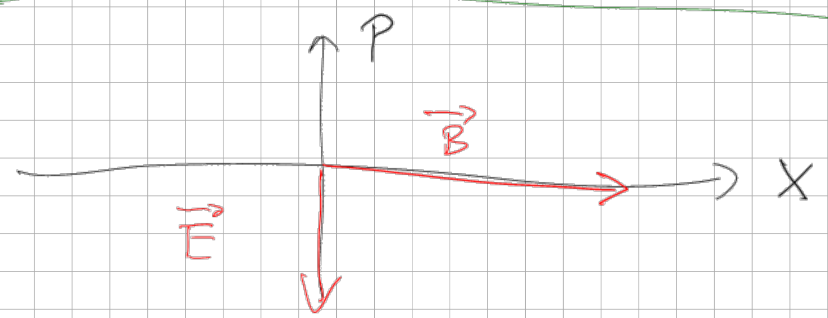
$$\hat{x}(t) = \frac{1}{\sqrt{2}}\left(\hat{a}\, e^{-i\omega t} + \hat{a}^+ e^{i\omega t}\right) = \hat{X}_{\omega t} \sim \hat{\vec{B}}$$

natural units: $\hbar = m = 1$

$$-\hat{p}(t) = \frac{1}{\sqrt{2}\,i}\left(\hat{a}\, e^{-i\omega t} - \hat{a}^+ e^{i\omega t}\right) = \hat{X}_{\omega t - \frac{\pi}{2}} \sim \hat{\vec{E}}$$

key point for phase space representation in quantum optics

$$\hat{\vec{E}} = -\frac{\partial \hat{\vec{A}}}{\partial t} = -\hat{\vec{\pi}}$$ (see Chapter 2)



Commutator:

$$[\hat{X}_{\Theta}, \hat{X}_{\Theta'}]_- = \dots = -i \sin(\Theta - \Theta')$$

$\Theta - \Theta' = -\frac{\pi}{2}$: $\quad [\hat{X}_{\Theta}, \hat{X}_{\Theta + \frac{\pi}{2}}]_- = i$

→ quadratures!

Example: $\hat{x}(t) = \hat{X}_{\omega t}$, $\hat{p}(t) = \hat{X}_{\omega t + \frac{\pi}{2}}$ harmonic oscillator

Two operators: $\hat{A}, \hat{B}$ ⇒ Heisenberg uncertainty relation (Appendix D)

$$\langle (\Delta \hat{A})^2 \rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle$$

$$\langle (\Delta \hat{B})^2 \rangle = \langle \hat{B}^2 \rangle - \langle \hat{B} \rangle^2 = \langle (\hat{B} - \langle \hat{B} \rangle)^2 \rangle$$

$$\langle(\Delta\hat{A})^2\rangle\cdot\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}\left|\langle[\hat{A},\hat{B}]_-\rangle\right|^2$$

Apply to $\hat{A}=\hat{x}_\Theta$, $\hat{B}=\hat{x}_{\Theta'}$

$$\langle(\Delta\hat{x}_\Theta)^2\rangle\langle(\Delta\hat{x}_{\Theta'})^2\rangle = \frac{1}{4}\left|\langle[\hat{x}_\Theta,\hat{x}_{\Theta'}]_-\rangle\right|^2 = \frac{1}{4}\sin^2(\Theta-\Theta') \leftarrow$$ valid for any state

largest possible uncertainty: $\Theta-\Theta' = -\frac{\pi}{2}$, i.e. for two quadratures

$$\Rightarrow \langle(\Delta\hat{x}_\Theta)^2\rangle\langle(\Delta\hat{x}_{\Theta+\frac{\pi}{2}})^2\rangle \underset{\text{Heisenberg}}{\geq} \frac{1}{4}$$

specialize to Fock state: $|\psi\rangle = |n\rangle$

$$\langle\hat{x}_\Theta\rangle_n = \frac{1}{\sqrt{2}}\langle n|\hat{a}e^{-i\Theta}+\hat{a}^+e^{i\Theta}|n\rangle \equiv 0$$

$$\langle\hat{x}_\Theta\hat{x}_{\Theta'}\rangle_n = \text{---} = n\cos(\Theta-\Theta') + \frac{1}{2}e^{-i(\Theta-\Theta')}$$

special case: $\Theta'=\Theta \Rightarrow \langle\hat{x}_\Theta^2\rangle_n = \langle(\Delta\hat{x}_\Theta)^2\rangle_n = n+\frac{1}{2}$

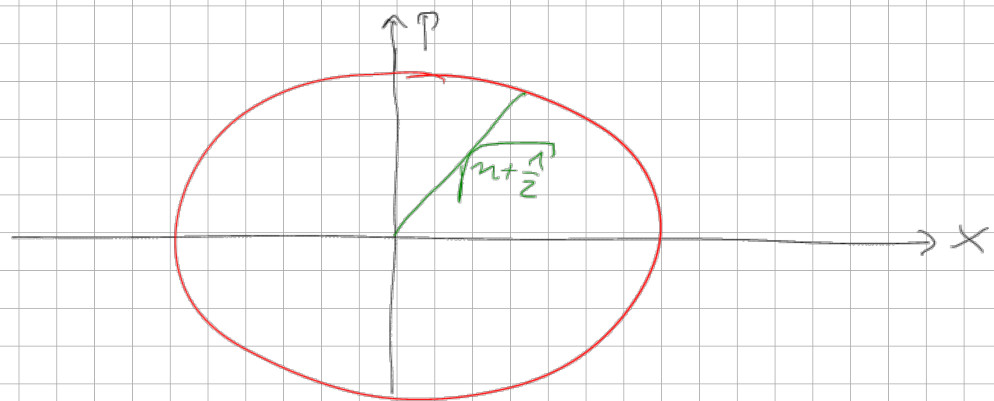
independent of Θ: holds ~~for~~ both $\hat{x}=\hat{x}_{\omega t}$ and $\hat{p}=\hat{x}_{\omega t+\frac{\pi}{2}}$

$$\langle(\Delta\hat{x})^2\rangle_n\cdot\langle(\Delta\hat{p})^2\rangle_n = \left(n+\frac{1}{2}\right)^2 \geq \frac{1}{4}$$

minimal uncertainty only for vacuum $n=0$

Phase space representation of Fock states:

- fixed particle number n
- circle around origin with radius $\sqrt{n+\frac{1}{2}}$



3.4 Unitary Transformation:

generic tool to generate other states
$\Rightarrow$ apply unitary transformation to Fock state

unitary: $\hat{U}^+ \hat{U} = \hat{U}\hat{U}^+ = 1 \Rightarrow \hat{U}^+ = \hat{U}^{-1}$

apply this to Fock state:

$$|n\rangle' = \hat{U}|n\rangle \quad \Leftrightarrow \quad |n\rangle = \hat{U}^+ |n\rangle'$$

action of $\hat{U}$ on some operator $\hat{O}$

$$'\langle n|\hat{O}'|m\rangle' = \langle n|\hat{U}^+\hat{O}'\hat{U}|m\rangle = \langle n|\hat{O}|m\rangle$$

$$\Rightarrow \hat{O} = \hat{U}^+\hat{O}'\hat{U} \quad \Leftrightarrow \quad \hat{O}' = \hat{U}\hat{a}\hat{U}^+$$

annihilation/creation operators

$$\hat{a}^{+\prime} = \hat{U}\hat{a}^+\hat{U}^+, \qquad \hat{a}' = \hat{U}\hat{a}\hat{U}^+$$

Canonical commutation relations do not change:

$$[\hat{a}', \hat{a}']_- = 0 = [\hat{a}^{+\prime}, \hat{a}^{+\prime}]_-, \quad [\hat{a}', \hat{a}^{+\prime}]_- = \hat{U}[\hat{a}, \hat{a}^+]_-\hat{U}^+ = 1$$

Transformed annihilation/creation operators also represent annihilation operators

$$\hat{n}'|n\rangle' = n|n\rangle' \qquad \hat{n}' = \hat{a}^{+\prime}\hat{a}'$$

$$\hat{a}^{+\prime}|n\rangle' = \sqrt{n+1}\,|n+1\rangle'$$

$$\hat{a}'|n\rangle' = \sqrt{n}\,|n-1\rangle'$$

Also transformed Fock states form a basis of the underlying Hilbert space

e.g. $\hat{a}'|0'\rangle = 0, \quad \langle 0|\hat{a}^{+\prime} = 0$

Note 1: Algebraic relations between transformed states and operators are identical to original states and operators

Note 2: Properties of transformed states $|n\rangle'$ can differ significantly from those of original states

Note 3: depends on concrete choice $\hat{U}$

example 1
coherent states
Heisenberg – Weyl Lie algebra h_4

example 2
squeezed states
$su(1,1)$
special unitary Lie algebra