Lie Algebra Methods:

1. Lie Algebra:

= vector space $\mathcal{G}$ is equipped with an inner product, here the commutator

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$$
$$(\hat{A}, \hat{B}) \longrightarrow [\hat{A}, \hat{B}]_-$$

Finite-dimensional vector space $\mathcal{G}$: $\hat{O}_1, \dots, \hat{O}_N$ are basis operators (= generators). And therefore $[\hat{O}_i, \hat{O}_j]_- = \sum_{k=1}^{N} \underbrace{c_{ijk}}_{\text{structure constants}} \hat{O}_k$

Two properties:

- $[\hat{O}_i, \hat{O}_j]_- = -[\hat{O}_j, \hat{O}_i]_- \Rightarrow c_{ijk} = -c_{jik}$
- Jacobi identity: $[\hat{A}, [\hat{B}, \hat{C}]_-]_- + [\hat{B}, [\hat{C}, \hat{A}]]_- + [\hat{C}, [\hat{A}, \hat{B}]]_- = 0$

$$\Rightarrow \quad c_{ijk} + c_{jki} + c_{kij} = 0$$

2. Similarity Transformations:

Let $\hat{S}$ be an invertible operator, so $\hat{S}^{-1}$ exists with $\hat{S}\hat{S}^{-1} = \hat{S}^{-1}\hat{S} = 1$

similarity transformation: $\hat{A} \longrightarrow \hat{S}\hat{A}\hat{S}^{-1}$, $\underline{\hat{O}_i} \longrightarrow \underline{\hat{S}\hat{O}_i\hat{S}^{-1}}$; $i = 1, \dots, N$

$$[\underline{\hat{S}\hat{O}_i\hat{S}^{-1}}, \underline{\hat{S}\hat{O}_j\hat{S}^{-1}}]_- = \hat{S}[\hat{O}_i, \hat{O}_j]_-\hat{S}^{-1} = \sum_{k=1}^{N} c_{ijk}\, \underline{\hat{S}\hat{O}_k\hat{S}^{-1}}$$

same structure constants

Let $f(z)$ be an analytic function: $f(z) = \sum_{k=0}^{\infty} f_k z^k$

$$\hat{S}\, f(\hat{A})\, \hat{S}^{-1} = \sum_{k=0}^{\infty} f_k \underbrace{\hat{S}\hat{A}^k S^{-1}}_{\hat{S}\hat{A}\underbrace{S^{-1}\hat{S}}_{=1}\hat{A}\underbrace{S^{-1}\hat{S}}_{=1}\hat{A}S^{-1} \ldots \hat{S}^{-1}} = \sum_{k=0}^{\infty} f_k \left(\hat{S}\hat{A}S^{-1}\right)^k = f(\hat{S}\hat{A}\hat{S}^{-1})$$

In particular in quantum optics, one uses similarity transformations of following type:

$$\hat{O}_i(t) = e^{t\hat{Z}}\, \hat{O}_i\, e^{-t\hat{Z}} \qquad \hat{Z} = \sum_{i=1}^{N} z_i \hat{O}_i$$

$$\frac{d\,\hat{O}_i(t)}{dt} = e^{t\hat{Z}} \underbrace{\left(\hat{Z}\hat{O}_i - \hat{O}_i\hat{Z}\right)}_{=[\hat{Z},\hat{O}_i]_-} e^{-t\hat{Z}} = \sum_{j=1}^{N} z_j\, e^{t\hat{Z}} \underbrace{[\hat{O}_j, \hat{O}_i]_-}_{=\sum_{k=1}^{N} c_{jik} \hat{O}_k} e^{-t\hat{Z}}$$

$$= \sum_{j=1}^{N} z_j c_{jik} \underbrace{e^{t\hat{Z}}\hat{O}_k e^{-t\hat{Z}}}_{=\hat{O}_k(t)}$$

$\Rightarrow$ set of N linear operator-valued differential equations

$\rightarrow$ has to be solved with "initial conditions"

$$\hat{O}_i(0) = \hat{O}_i \quad ; \quad i = 1, \ldots, N$$

3. Disentangling an exponential:

$$\hat{Z} = \sum_{i=1}^{N} z_i \hat{O}_i, \qquad e^{t\hat{Z}} = e^{f_1(t)\hat{O}_1} \cdot e^{f_2(t)\hat{O}_2} \cdot \ldots \cdot e^{f_N(t)\hat{O}_N}$$

$f_1(t), f_2(t), \ldots, f_N(t)$ have to be determined

$$(\hat{A}\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1}$$

Method: differentiate with respect to t:

$$\underbrace{\left(\frac{d}{dt} e^{t\hat{Z}}\right) e^{-t\hat{Z}}}_{e^{t\hat{Z}}\,\hat{Z}\,e^{-t\hat{Z}} = \hat{Z}} = \frac{d}{dt}\left(e^{g_1(t)\hat{O}_1} e^{g_2(t)\hat{O}_2} \cdots e^{g_N(t)\hat{O}_N}\right) e^{-g_N(t)\hat{O}_N} \cdots e^{-g_2(t)\hat{O}_2} e^{-g_1(t)\hat{O}_1}$$

$$\hat{Z} = \sum_{i=1}^{N} z_i \hat{O}_i = \dot{g}_1(t)\,\hat{O}_1 + \dot{g}_2(t)\left\{e^{g_1(t)\hat{O}_1}\,\hat{O}_2\,e^{-g_1(t)\hat{O}_1}\right\}$$

$$+ \dot{g}_3(t)\left\{e^{g_1(t)\hat{O}_1}\left\{e^{g_2(t)\hat{O}_2}\,\hat{O}_3\,e^{-g_2(t)\hat{O}_2}\right\}e^{-g_1(t)\hat{O}_1}\right\} + \cdots$$

- similarity transformations on right-hand side have to be evaluated one by one
- left- and right side are expansions into generators

$\longrightarrow$ equate respective coefficients

$\longrightarrow$ N nonlinear differential equations for $g_1(t), g_2(t), \ldots, g_N(t)$

solved with $g_1(0) = g_2(0) = \cdots = g_N(0) = 0$

4. Coherent States:

Heisenberg – Weyl algebra $\overbrace{h_4}^{N=4} = \{1, \hat{a}, \hat{a}^+, \hat{a}^+\hat{a} = \hat{n}\}$

$[\hat{a}, \hat{a}^+]_- = 1$, $[\hat{n}, \hat{a}^+]_- = \hat{a}^+$, $[\hat{n}, \hat{a}]_- = -\hat{a}$ non-vanishing commutator relations

$\Rightarrow$ operator algebra of harmonic oscillator

5. Squeezed States:

Lie algebra $\underbrace{su}_{\text{special unitary}}(1,1) = \{\hat{K}_1, \hat{K}_2, \hat{K}_3\}$

$[\hat{K}_1, \hat{K}_2]_- = -i\hat{K}_3$, $[\hat{K}_2, \hat{K}_3]_- = i\hat{K}_1$, $[\hat{K}_3, \hat{K}_1]_- = i\hat{K}_2$

angular momentum algebra $su(2) = \{\hat{L}_1, \hat{L}_2, \hat{L}_3\}$

$[\hat{L}_1, \hat{L}_2]_- = i\hat{L}_3, \quad [\hat{L}_2, \hat{L}_3]_- = i\hat{L}_1, \quad [\hat{L}_3, \hat{L}_1]_- = i\hat{L}_2$

useful: $\hat{L}_\pm = \hat{L}_1 \pm i\hat{L}_2$

$[\hat{L}_+, \hat{L}_-]_- = 2\hat{L}_3, \quad [\hat{L}_3, \hat{L}_\pm]_- = \pm\hat{L}_\pm$

in analogy: $\hat{K}_\pm = \hat{K}_1 \pm i\hat{K}_2$

$[\hat{K}_+, \hat{K}_-] = -i\hat{K}_3, \quad [\hat{K}_3, \hat{K}_\pm]_- = \pm\hat{K}_\pm$

Using the harmonic oscillator operators we find:

$$\hat{K}_+ = \frac{\hat{a}^{+2}}{2}, \quad \hat{K}_- = \frac{\hat{a}^2}{2}, \quad \hat{K}_3 = \frac{1}{2}(\underbrace{\hat{n}}_{=\hat{a}^+\hat{a}} + \frac{1}{2})$$

homework

3.5 Shifting Operator:

transformation: $\hat{D}(\alpha) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}, \quad \alpha \in \mathbb{C}$

$$\hat{D}^+(\alpha) = e^{\alpha^*\hat{a} - \alpha\hat{a}^+} = e^{-(\alpha\hat{a}^+ - \alpha^*\hat{a})} = \hat{D}(-\alpha)$$

$$\hat{D}(\alpha)\hat{D}(-\alpha) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}\, e^{-(\alpha\hat{a} - \alpha^*\hat{a})} = 1 = \hat{D}(0) \Rightarrow \hat{D}^+(\alpha) = \hat{D}(-\alpha) = \hat{D}(\alpha)^{-1}$$

transformed annihilation/creation operator

$$\hat{a}'(\alpha) = \hat{D}(\alpha)\,\hat{a}\,\underbrace{\hat{D}^+(\alpha)}_{=\hat{D}(-\alpha)}$$

Hint: $\hat{a}'(\alpha t) = e^{t(\alpha\hat{a}^+ - \alpha^*\hat{a})}\,\hat{a}\,e^{-t(\alpha\hat{a}^+ - \alpha^*\hat{a})}$

- differentiate with respect to t
- solve operator-valued differential equation
- initial condition: $\hat{a}'(0) = \hat{a}$

$$\underbrace{\hat{D}(\alpha t)}_{\hat{a}'(\alpha t)} \overset{t=1}{=} \underbrace{\hat{D}(\alpha)}_{\hat{a}'(\alpha)} = \hat{a} - \alpha$$

(shifting operator)

$|\alpha\rangle := |0\rangle' = \hat{D}(\alpha)|0\rangle$

$\underbrace{\hat{a}'(\alpha)}|\alpha\rangle = \hat{a}'(\alpha)|0\rangle' \equiv 0$

$(\hat{a}-\alpha)|\alpha\rangle = 0 \quad \rightarrow \quad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$\hat{a}^+|\alpha\rangle = \alpha^*|\alpha\rangle$

coherent state $|\alpha\rangle$ is eigenstate of annihilation operator to eigenvalue α

introduced by Roy Glauber in 1963

3.6 Group Property:

$$\hat{D}(\alpha)\hat{D}(\beta) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}\, e^{\beta\hat{a}^+ - \beta^*\hat{a}} = \ldots = \hat{D}(\alpha+\beta)\, e^{i\,\mathrm{Im}(\alpha\beta^*)}$$

- $\mathrm{Im}(\alpha\beta^*) = 0$: $\alpha \rightarrow \hat{D}(\alpha)$ is a representation of addition of complex numbers in space of unitary transformation
- $\mathrm{Im}(\alpha\beta^*) \neq 0$: projective representation

3.7 Properties of Coherent States:

$\langle \cdot \rangle_\alpha = \langle\alpha|\cdot|\alpha\rangle$

$$\langle \hat{x}_\Theta\rangle_\alpha = \langle\alpha|\hat{x}_\Theta|\alpha\rangle = \frac{1}{\sqrt{2}}\langle\alpha|\hat{a}e^{-i\Theta} + \hat{a}^+e^{+i\Theta}|\alpha\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\alpha e^{-i\Theta} + \alpha^* e^{i\Theta}\right) = \sqrt{2}\left(\mathrm{Re}\,\alpha\cos\Theta + \mathrm{Im}\,\alpha\sin\Theta\right)$$

1. Electromagnetic field:

$$\langle\hat{\vec{E}}(\vec{x},t)\rangle_\alpha = 2E_0|\alpha|\vec{\varepsilon}\sin(\omega t-\varphi)\sin(k_x x) \quad ; \quad |\alpha| = \sqrt{(\mathrm{Re}\,\alpha)^2 + (\mathrm{Im}\,\alpha)^2}$$

$$\langle\hat{\vec{B}}(\vec{x},t)\rangle_\alpha = \frac{2E_0}{c}|\alpha|\vec{e}_x\times\vec{\varepsilon}\cos(\omega t-\varphi)\cos(k_x x); \quad \varphi = \arctan\frac{\mathrm{Im}\,\alpha}{\mathrm{Re}\,\alpha}$$

$\Rightarrow$ classical standing wave in cavity

$\Rightarrow$ coherent states are useful to represent laser fields

$$\langle \hat n \rangle_\alpha = \langle\alpha|\hat a^\dagger \hat a|\alpha\rangle = |\alpha|^2 \Rightarrow |\alpha| = \sqrt{\langle \hat n\rangle_\alpha} \qquad \Rightarrow \langle \hat{\vec E}\rangle_\alpha \sim \sqrt{\langle \hat n\rangle_\alpha}$$

2 harmonic oscillator:

$$\begin{pmatrix} \langle \hat x(t)\rangle_\alpha \\ \langle \hat p(t)\rangle_\alpha \end{pmatrix} = \sqrt{2}\begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} \mathrm{Re}\,\alpha \\ \mathrm{Im}\,\alpha \end{pmatrix}$$

$$\Rightarrow \mathrm{Re}\,\alpha = \frac{1}{\sqrt2}\langle \hat x(0)\rangle_\alpha, \quad \mathrm{Im}\,\alpha = \frac{1}{\sqrt2}\langle \hat p(0)\rangle_\alpha$$

$\Rightarrow$ α is determined from initial condition

Second moment:

$$\langle \hat x(\theta)\,\hat x(\theta')\rangle_\alpha = \dots = \langle \hat x(\theta)\rangle_\alpha \langle \hat x(\theta')\rangle_\alpha + \frac{1}{2} e^{-i(\theta-\theta')}$$

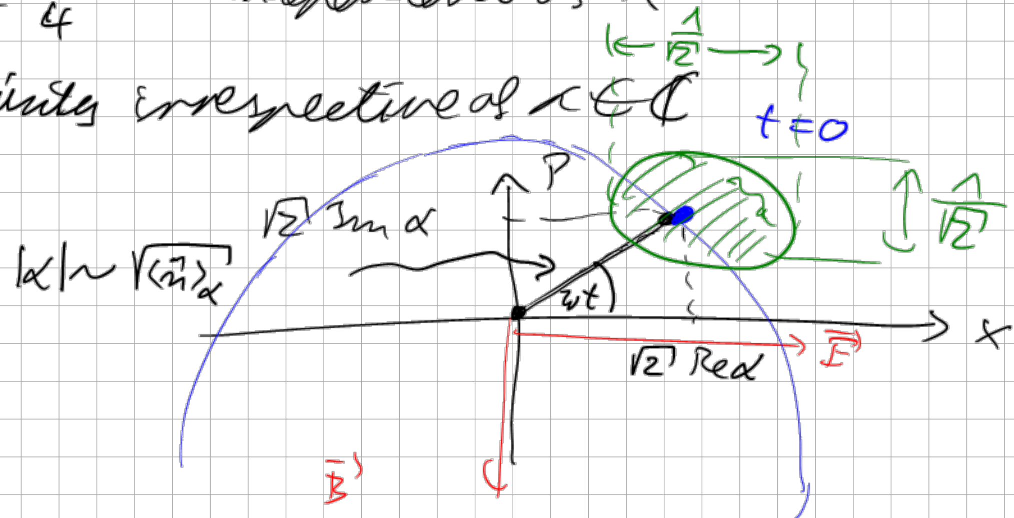
$$\langle (\Delta \hat x(\theta))^2\rangle_\alpha = \langle \hat x(\theta)^2\rangle_\alpha - \langle \hat x(\theta)\rangle_\alpha^2 = \frac{1}{2}$$ independent of θ

Check for Heisenberg:

$$\langle (\Delta \hat x(\theta))^2\rangle_\alpha \cdot \langle (\Delta \hat x(\theta - \tfrac{\pi}{2}))^2\rangle_\alpha \overset{(\geq)}{=} \frac{1}{4}$$ independent of α

coherent states have minimal uncertainty irrespective of $\alpha \in \mathbb{C}$

phase space representation of a coherent state



3.8 Squeezed States:

Idea: Reduce (enhance) quantum noise in one of the two quadratures

$$\hat{S}(\xi) = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})} \qquad \xi \in \mathbb{C}$$

nonlinearity!

$$\hat{S}^+(\xi) = e^{\frac{1}{2}(\xi \hat{a}^{+2} - \xi^* \hat{a}^2)} = \hat{S}(-\xi)$$

$$\hat{S}^+(\xi)\,\hat{S}(\xi) = \hat{S}(-\xi)\hat{S}(\xi) = \hat{S}(0) = 1 \implies \hat{S}^+(\xi) = \hat{S}^{-1}(\xi)$$

transformed annihilation/creation operators

$$\hat{a}'(\xi) = \hat{S}(\xi)\,\hat{a}\,\hat{S}^+(\xi)$$ has to be determined

Consider $$\hat{a}'(\xi t) = e^{\frac{t}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})}\,\hat{a}\,e^{-\frac{t}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})}$$

→ differentiate in t, one derivative not enough

→ initial condition: $\hat{a}'(0) = \hat{a}$