Last Time:

Field operator: $\hat{\vec{A}}(\vec{x},t)$

- Homogeneous wave equation: $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\hat{\vec{A}}(\vec{x},t) = \vec{0}$ (*)
- Coulomb gauge: $\operatorname{div} \vec{A}(\vec{x},t) = 0$

2.8 Solution of Wave Equation:

$$\hat{\vec{A}}(\vec{x},t) = \int d^3k\, \hat{\vec{A}}(\vec{k},t)\, e^{i\vec{k}\vec{x}}$$

$\longrightarrow$ inserting in (1)

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \underbrace{\vec{k}^2 c^2}_{=\omega_{\vec{k}}^2}\right)\hat{\vec{A}}(\vec{k},t) = \vec{0}$$

linear dispersion

harmonic oscillator equation:

$$\hat{\vec{A}}(\vec{k},t) = \hat{\vec{A}}^{(1)}(\vec{k})\, e^{-i\omega_{\vec{k}}t} + \hat{\vec{A}}^{(2)}(\vec{k},t)$$

$e^{+i\omega_{\vec{k}}t}$

$$\hat{\vec A}(\vec x,t)=\int d^3k\left\{\hat{\vec A}^{(1)}(\vec k)\, e^{i(\vec k\vec x-\omega_{\vec k}t)}+\underbrace{\hat{\vec A}^{(2)}(\vec k)\, e^{i(\vec k\vec x+\omega_{\vec k}t)}}_{\hat{\vec A}^{(2)}(-\vec k)\, e^{-i(\vec k\vec x-\omega_{\vec k}t)}}\right\}$$

$$\vec k\to-\vec k$$

$$\omega_{-\vec k}=\omega_{\vec k}$$

real vector potential: $\vec A(\vec x,t)=\vec A^*(\vec x,t)$

field operator has to be self-adjoint: $\hat{\vec A}(\vec x,t)=\hat{\vec A}^+(\vec x,t)$

$$\hat{\vec A}^+(\vec x,t)=\int d^3k\left\{\hat{\vec A}^{(1)+}(\vec k)\, e^{-i(\vec k\vec x-\omega_{\vec k}t)}+\hat{\vec A}^{(2)+}(-\vec k)\, e^{+i(\vec k\vec x-\omega_{\vec k}t)}\right\}$$

$$\Rightarrow\ \hat{\vec A}^{(1)}(\vec k)=\hat{\vec A}^{(2)+}(-\vec k)\ \Leftrightarrow\ \hat{\vec A}^{(1)+}(\vec k)=\hat{\vec A}^{(2)}(-\vec k)$$

one degree of freedom is eliminated:

$$\hat{\vec A}(\vec k):=\hat{\vec A}^{(1)}(\vec k),\quad \hat{\vec A}^{(2)}(\vec k)=\hat{\vec A}^+(-\vec k)$$

$$\hat{\vec A}(\vec x,t)=\int d^3k\left\{\underbrace{\hat{\vec A}(\vec k)}_{?}\, e^{i(\vec k\vec x-\omega_{\vec k}t)}+\hat{\vec A}^+(\vec k)\, e^{-i(\vec k\vec x-\omega_{\vec k}t)}\right\}$$

2.8 Polarization Vectors:

step back: more detailed understanding of plane waves

consider two plane waves with $\vec{k}$, $\omega_{\vec{k}} = c|\vec{k}|$:

$\vec{A}_1(\vec{x},t) = A_1 \, \vec{\varepsilon}_1 \, e^{i(\vec{k}\vec{x} - \omega_{\vec{k}} t)}$ $\quad \vec{A}_2(\vec{x},t) = A_2 \, \vec{\varepsilon}_2 \, e^{i(\vec{k}\vec{x} - \omega_{\vec{k}} t)}$

linear polarized plane waves → unique polarization

A_i: complex amplitudes, $\quad \vec{\varepsilon}_i$: polarization vectors

orthonormal: $\vec{\varepsilon}_i^{\,*} \cdot \vec{\varepsilon}_j = \delta_{ij}$

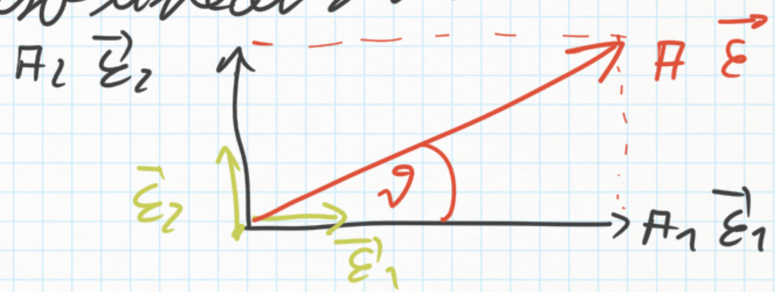
$\vec{A}(\vec{x},t) = \vec{A}_1(\vec{x},t) + \vec{A}_2(\vec{x},t) = (A_1 \vec{\varepsilon}_1 + A_2 \vec{\varepsilon}_2)\, e^{i(\vec{k}\vec{x} - \omega_{\vec{k}} t)}$

general case: $\quad A_1 = |A_1|\, e^{i\varphi_1}, \quad A_2 = |A_2|\, e^{i\varphi_2}$

special case: $\underline{\varphi_1 = \varphi_2 = \varphi}$ ⇒ sum is also linear polarization

$\Rightarrow \vec{A}(\vec{x},t) = A \, \vec{\varepsilon} \, e^{i(\vec{k}\vec{x} - \omega_{\vec{k}} t)}$

$A = \sqrt{A_1^2 + A_2^2}, \quad \vartheta = \arctan \frac{A_2}{A_1}$



$\varphi_1 \neq \varphi_2$: elliptically polarised plane wave

Illustration: circularly polarised " "

$$A_1 = \frac{A_0}{\sqrt{2}}, \quad A_2 = \pm i \frac{A_0}{\sqrt{2}} \; ; \quad \varphi_2 - \varphi_1 = \pm \frac{\pi}{2}$$

$$\Rightarrow \vec{A}(\vec{x},t) = \frac{A_0}{\sqrt{2}} \left(\vec{\varepsilon}_1 \pm i \; \vec{\varepsilon}_2 \right) e^{i(\vec{k}\vec{x} - \omega_{\vec{k}} t)}$$

concrete:

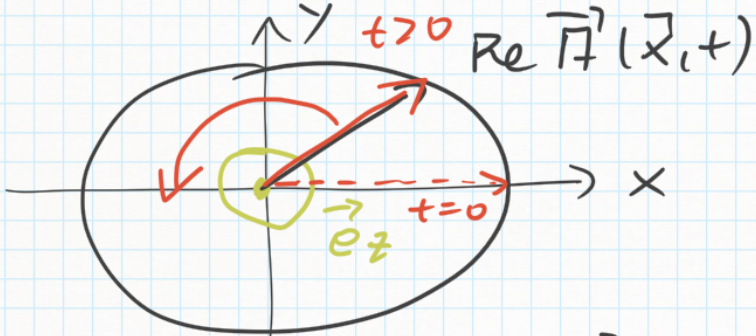
$$\underbrace{\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{propagation in } z\text{-direction}}, \quad \underbrace{\vec{\varepsilon}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\varepsilon}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{polarization in } xy\text{-plane}}$$

$$\vec{A}(\vec{x},t) = \frac{A_0}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} e^{i \underbrace{(k \vec{e}_z \vec{x} - \omega_k \vec{e}_z t)}_{= k(z - ct)}}$$

$\rightarrow$ real part is physical

$$\mathrm{Re}\,\vec{A}(\vec{x},t) = \frac{A_0}{\sqrt{2}} \begin{pmatrix} \cos k(\cancel{z} - ct) \\ \mp \sin k(\underset{0}{\cancel{z}} - ct) \end{pmatrix}$$

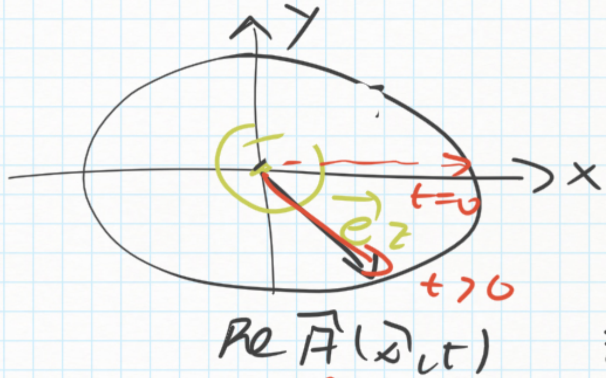
upper sign



z=0

anti-clockwise

lower sign



z=0

clockwise

observed for somebody looking in the direction of the incoming laser beam

optics: left-circular polarized

elementary particle physics: positive helicity

right-circular polarized

negative helicity

2.4 Construct of Polarization Vectors:

Aim: define helicity of an electromagnetic wave

Motivation: characterization properties of a photon

spin 1 $\hat{=}$ internal angular momentum

rest mass $= 0$

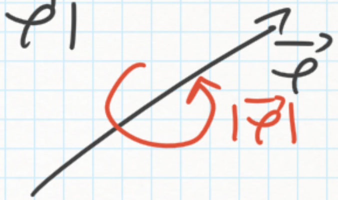
positive helicity

negative helicity

Method: representation theory of rotations

most general rotation matrix:

$$R_{jk}(\vec\varphi) = \frac{\varphi_i}{|\vec\varphi|}\varepsilon_{ikj}\sin|\vec\varphi| + \frac{\varphi_j\varphi_k}{|\vec\varphi|^2}(1-\cos|\vec\varphi|) + \delta_{jk}\cos|\vec\varphi|$$



- Rotation axis $\vec\varphi$: $\quad R_{jk}(\vec\varphi)\,\varphi_k = (+1)\cdot\varphi_j$ ✓
- Rotation angle $|\vec\varphi|$:

$$\mathrm{Tr}\,R(\vec\varphi) = R_{jj}(\vec\varphi) = 1-\cos|\vec\varphi| + 3\cdot\cos|\vec\varphi| = 1+2\cos|\vec\varphi| \quad ✓$$

infinitesimal rotation

Lie theorem

$$R(\vec\varphi) = \underbrace{e^{-i\,\vec\varphi\cdot\underbrace{\vec S}_{3\times 3\ \text{matrix}}}}_{3\times 3\ \text{matrix}}$$

generators of rotations = spin $\vec S$

$$(S_m)_{jk} = i\,\left.\frac{\partial R_{jk}(\vec\varphi)}{\partial\varphi_m}\right|_{\vec\varphi=\vec 0}$$

$$S_1 = i\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\quad S_2 = i\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\quad S_3 = i\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}$$

$$\Rightarrow \quad [S_i, S_j]_- = i\,\varepsilon_{ijk}\,S_k$$

angular momentum algebra

definition helicity:

$$\hat h(\vec k) = \frac{\vec k}{|\vec k|}\cdot \vec S = k_x S_1 + k_y S_2 + k_z S_z = \frac{i}{k}\begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix}$$

polarization vector $\vec\varepsilon(\vec k, \lambda)$

helicity $\lambda = \pm 1$

$$\vec A(\vec x, t) = A\,\vec\varepsilon(\vec k, \lambda)\, e^{i(\vec k\vec x - \omega_k t)}$$

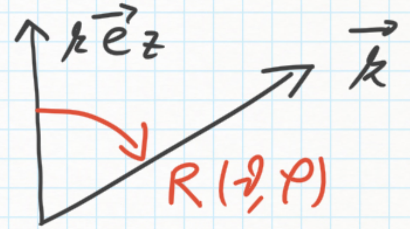
$$\Rightarrow \quad \boxed{\hat h(\vec k)\,\vec\varepsilon(\vec k, \lambda) = \lambda\,\vec\varepsilon(\vec k, \lambda)}$$

Example: $\vec\varepsilon(k\vec e_z, \lambda) = \frac{1}{\sqrt2}\begin{pmatrix} 1 \\ \lambda i \\ 0 \end{pmatrix}$; $\hat h(k\vec e_z) = S_3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\hat h(k\vec e_z)\,\vec\varepsilon(k\vec e_z, \lambda) = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \lambda i \\ 0 \end{pmatrix}\frac{1}{\sqrt2} = \lambda\,\vec\varepsilon(k\vec e_z, \lambda)$$

$$\vec k = k\begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} = \underbrace{R(\vartheta, \varphi)}\, k\vec e_z$$

$$= R_z(\varphi)\; R_y(\vartheta)$$



$$R_z(\varphi) = e^{-iS_3\varphi} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\vartheta) = e^{-iS_2\vartheta} = \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix}$$

$$R(\vartheta,\varphi) = R_z(\varphi)R_y(\vartheta) = \begin{pmatrix} \cos\vartheta\cos\varphi & -\sin\varphi & \boxed{\sin\vartheta\cos\varphi} \\ \cos\vartheta\sin\varphi & \cos\varphi & \boxed{\sin\vartheta\sin\varphi} \\ -\sin\vartheta & 0 & \boxed{\cos\vartheta} \end{pmatrix}$$

$$\vec{k} = \underline{R(\vartheta,\varphi)}\, k\,\vec{e}_z \;\hat{=}\; \vec{\varepsilon}(\vec{k},\lambda) = \underline{R(\vartheta,\varphi)}\,\vec{\varepsilon}(k\vec{e}_z,\lambda)$$

exercise

$$\Rightarrow \vec{\varepsilon}(\vec{k},\lambda) \overset{!}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\vartheta\cos\varphi - \lambda i \sin\varphi \\ \cos\vartheta\sin\varphi + \lambda i \cos\varphi \\ -\sin\vartheta \end{pmatrix}$$

$$\hat{h}(\vec{k})\,\vec{\varepsilon}(\vec{k},\lambda) = \lambda\,\vec{\varepsilon}(\vec{k},\lambda) \quad ✓$$

$$\vec{\varepsilon}(\vec{k},\lambda)\Big|_{\vec{k}=k\vec{e}_z} \xlongequal{\vartheta=0,\ \varphi=0} \vec{\varepsilon}(k\vec{e}_z,\lambda) \quad ✓$$